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Managing Climate Change Under Uncertainty: Recursive Integrated Assessment at an Inflection Point

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Abstract
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Economists view climate change as resulting from a massive market failure: today's greenhouse gas emissions generate global warming that will affect people and ecosystems for many generations to come, yet those emissions often carry a market price of zero. Economists therefore often view the primary goal of climate change policy as pricing greenhouse gas emissions to account for the costs of the climate change they generate. In a valiant attempt to quantify these costs, economists have developed integrated assessment models (IAMs) that couple climatic and economic modules.[1] These models determine the optimal emission price by trading off the benefits of allowing more emissions today against the cost of future climate change. However, accurately quantifying the costs of future climate change is an impossible task: these costs depend, among much else, on the uncertain unfolding of climate change, on the uncertain consequences of climate change for the economy and for wellbeing, and on the uncertain evolution of technology and the economy. Conventional IAMs can provide insight into which parameters are likely to be important for the optimal emission price, but their deterministic structure can only take one so far in a world of uncertainty.

A newer generation of IAMs incorporates uncertainty into the optimal emission price. These "recursive" IAMs solve a dynamic programming version of a standard IAM. Their modeled policymaker is therefore cognizant of uncertainty and also of the possibility of learning through future observations of the climate and the economy. In principle, the recursive approach to IAMs can incorporate all of the uncertainties that bedevil the application of standard IAMs to policy, though in practice many of these uncertainties are of such a "deep" nature that they are difficult to formalize.

The present review outlines the recursive approach to climate change, demonstrates how to use recursive modeling to generate deeper theoretical insight into the drivers of optimal emission policy, and summarizes the main conclusions of the recent literature. We judge this literature to be at an inflection point. The literature has made enormous strides from a standing start. However, the marginal benefit of yet another model incorporating one more standalone source of uncertainty is low. To date, most recursive models have been limited by the use of numerical methods that are advanced relative to much work in economics but nonetheless not well-adapted to the high-dimensional state spaces that characterize IAMs.[2] We propose that adopting computational methods even closer to the frontier of computational economics will allow recursive IAMs to be more detailed and representative of reality and thereby expand the types of questions that they can explore. To this end, we describe the standard numerical approach to solving a recursive IAM and provide a guide to some promising numerical methods that are not yet common in economics research. We hope that the coming second wave of recursive IAMs will use these methods to ask new types of questions.

[1] IAMs' estimates of the social cost of greenhouse gas emissions have formed the backbone of U.S. policy (Greenstone et al., 2011). See Nordhaus (2013) and Kelly and Kolstad (1999a) for overviews of climate-economy integrated assessment.

[2] IAMs' climate modules require several state variables on top of those required by their economic modules.

In order to motivate the recursive approach to policy under uncertainty, the next section formally describes the shortcomings of what some may see as the main alternative: the Monte Carlo approach to policy under uncertainty. Section 2 then formalizes the recursive approach. Section 3 decomposes the channels through which uncertainty about future warming affects optimal policy, and Section 4 quantifies these channels in a recursive extension of the benchmark DICE IAM (which we will make publicly available). We hope that these two sections provide future literature with a guide for using recursive IAMs to generate theoretical insight. Section 5 summarizes key conclusions from the first wave of research with recursive IAMs. Section 6 discusses numerical methods, both standard and frontier. Section 7 concludes with suggested directions for future research. The appendix extends the theoretical analysis to the case of Epstein-Zin-Weil preferences, provides the full equations for our recursive IAM, and describes best practices for validating solutions to recursive models.

1 The Shortcomings of the Monte Carlo Approach to Policy Under Uncertainty

For many years, economists numerically analyzed the implications of uncertainty for climate policy by undertaking Monte Carlo analyses of integrated assessment models (e.g., Pizer, 1999; Roughgarden and Schneider, 1999; Tol, 1999; Nordhaus, 2008). A newer, growing literature analyzes uncertainty by constructing recursive, dynamic programming versions of integrated assessment models. We begin by describing the Monte Carlo approach and its shortcomings for analyzing most questions of interest about the policy implications of uncertainty. We describe the dynamic programming approach in the next section.

Consider a stylized, simplified integrated assessment model. The policymaker controls emissions e_t, trading off current utility from the consumption enabled by additional emissions against the welfare loss from triggering additional warming in the future. The policymaker's time t per-period utility is $u_t(e_t; T_t)$, with utility increasing and concave in emissions ($\partial u_t/\partial e_t > 0, \partial^2 u_t/\partial e_t^2 < 0$) and decreasing in temperature ($\partial u_t/\partial T_t < 0$). We allow utility to depend on time to reflect the potential for exogenous growth in population, capital stocks, or technology. Temperature evolves as $T_{t+1} = f(T_t, e_t, \epsilon_{t+1}; s)$. Increasing time t emissions increases time $t+1$ temperature ($\partial f/\partial e_t \geq 0$), and increasing time t emissions can also increase temperature after time $t+1$ via the dependence of f on T_t. The parameter s controls the climate's responsiveness to emissions (commonly referred to as "climate sensitivity"), with $\partial^2 f/\partial e_t \partial s, \partial^2 f/\partial T_t \partial s \geq 0$. The integrated assessment modeler does not know the true value of s and is interested in the implications of uncertainty about s for welfare and for emission policy. The integrated assessment modeler estimates the mean of s to be μ_0. Finally, ϵ_{t+1} will be a random shock that will prevent the modeled policymaker from immediately learning the true value of s, but we begin by treating ϵ_{t+1} as deterministic so

as to simplify the exposition.

If we knew that s took on a particular value s_i with probability 1, then our policymaker could solve a deterministic problem. Specifically, the policymaker would solve:

$$W^{det}(T_0; s_i) = \max_{\{e_t\}_0^\tau} \sum_{t=0}^{\tau} \beta^t u_t(e_t; T_t), \quad \text{subject to } T_{t+1} = f(T_t, e_t, \epsilon_{t+1}; s_i),$$

where $\beta \in (0, 1)$ is the per-period discount factor, model time starts at time 0, and the policymaker's horizon extends out to $\tau > 0$, with τ potentially infinite. Solving this problem yields the policymaker's maximized welfare W^{det} as a function of initial temperature and s_i, and it also yields optimal policy $e_t^{det}(s_i)$ in every period.

Now consider a Monte Carlo analysis of uncertain s. A Monte Carlo analysis of policy under uncertainty solves the deterministic problem for several different values s_i and compares the weighted average welfare and policy over these simulations to the welfare and policy resulting from fixing $s_i = \mu_0$. Formally, Monte Carlo analyses commonly define maximized welfare under uncertainty as[3]

$$W^{MC}(T_0) \coloneqq E_0\left[W^{det}(T_0; s_i)\right] = \int_{-\infty}^{\infty} W^{det}(T_0; s_i)\, p(s_i)\ \mathrm{d}s_i,$$

where E_0 refers to expectations at time 0 and $p(s_i)$ is the modeler's probability that $s = s_i$. And Monte Carlo analyses commonly define policy under uncertainty as

$$e_t^{MC} \coloneqq \int_{-\infty}^{\infty} e_t^{det}(s_i)\, p(s_i)\ \mathrm{d}s_i.$$

Finally, Monte Carlo analyses commonly define the welfare cost of uncertainty as $W^{det}(T_0; \mu_0) - W^{MC}(T_0)$ and the effect of uncertainty on policy as $e_t^{MC} - e_t^{det}(\mu_0)$.

Consider the information structure of the Monte Carlo analysis. Who is it that is uncertain about s? Whose welfare is given by $W^{MC}(T_0)$? With respect to whose expectations is e_t^{MC} an average policy?

In the Monte Carlo analysis, the *modeled policymaker* is never uncertain about s. In every simulation, the policymaker knows that $s = s_i$. In particular, the policymaker is not uncertain about s when choosing emissions: emissions are chosen for a deterministic world. Welfare $W^{MC}(T_0)$ reflects the *modeler's* uncertainty about which model to run (i.e., which value of s_i to code), but it does not reflect uncertainty within a model about which value of s_i is the correct one. A Monte Carlo analysis therefore assumes that we do not know the true climate sensitivity today but will know it very soon, before we get around to formulating emission control policies.

[3]In practice, the integral is discretized or modelers use a discrete distribution for s. Many modelers just average over N values for s_i, in which case $p(s_i) = 1/N$.

Not only are W^{MC} and e^{MC} conceptually odd objects, but the differences $W^{det}(T_0;\mu_0) - W^{MC}(T_0)$ and $e_t^{MC} - e_t^{det}(\mu_0)$ may not even properly sign the effect of uncertainty on policy, as demonstrated in Crost and Traeger (2013). Further, Crost and Traeger (2013) show that a Monte Carlo analysis implies that uncertainty about a damage coefficient simultaneously reduces the "optimal" abatement rate and increases the "optimal" abatement cost, two results with seemingly contradictory implications for policy.

In sum, a Monte Carlo analysis can be useful for analyzing a model's sensitivity to assumptions, but a Monte Carlo analysis cannot answer many important questions about the implications of uncertainty for welfare and policy.

2 The Recursive Dynamic Programming Approach to Uncertainty

How can modelers handle uncertainty? We here describe the recursive dynamic programming approach to climate change.[4,5] The next section shows how to use the results of dynamic programming to study the implications of uncertainty.

Consider the same setting as in Section 1. Maximized welfare with uncertain s is

$$W^{unc}(T_0) \coloneqq \max_{\{e_t(T_t)\}_0^\tau} E_0\left[\sum_{t=0}^{\tau} \beta^t u_t(e_t;T_t)\right], \text{ subject to } T_{t+1} = f(T_t, e_t, \epsilon_{t+1}; s).$$

Let $e_t^{unc}(T_t)$ denote the optimized emissions level at time t and temperature T_t. Note the difference between this expression and the expression for $W^{MC}(T_0)$ from the previous section. Here, we formulate policy by maximizing over the expectation of utility in every period. There, we developed policy without using expectations at all, instead taking expectations over s only along policy trajectories that had already been optimized. Here, our policy $e_t^{unc}(T_t)$ depends on both time and temperature. There, our policy e_t^{MC} depended only on time because we averaged over policies which could each perfectly predict future tempera-

[4]A stochastic control approach can also properly address uncertainty, but it typically requires simplified types of uncertainty (such as two-point distributions) and/or shortened time horizons (such as two-period models). We here describe the benchmark approach to integrating uncertainty into a full climate-economy integrated assessment model. Recently, a literature following Golosov et al. (2014) has developed analytic IAMs that do not require advanced numerical methods. However, assumptions such as logarithmic utility combine to make the model effectively linear, which can make uncertainty uninteresting in these settings.

[5]Lemoine (2015) studies the implications of uncertainty without using an optimization-based approach, but he studies the implications for the social cost of carbon, not the optimal emission tax or the optimal quantity of emissions. The social cost of carbon is defined for a given emission trajectory, and it matches the optimal emission tax along the optimal emission trajectory.

ture.[6]

The standard way of solving the problem is to write down its recursive formulation, in the form of a Bellman equation.[7] Use $V_t(T_t)$ to denote the value of the optimal policy program at time t when temperature is T_t. Thus, $V_0(T_0) = W^{unc}(T_0)$. If the policymaker knows his best possible value at time $t+1$ for any given T_{t+1}, then he can select his optimal policy at time t without needing to consider policies beyond t. This insight yields the following relationship:[8]

$$V_t(T_t) = \max_{e_t} \left\{ u_t(e_t; T_t) + \beta E_t \left[V_{t+1}(T_{t+1}) \right] \right\}, \tag{1}$$

where T_{t+1} depends on T_t and e_t as before and where E_t denotes expectations at the time t information set. Note three features of this setup, as opposed to the Monte Carlo setup. First, we here develop the optimal time t policy for any potential realization of T_t. Second, the optimal policy here accounts for the fact that future realizations $T_{t+1}, T_{t+2}, ...$ are uncertain. Third, we use time t information when taking expectations over V_{t+1}, known as the continuation value. The time t policy can thus depend on information about s obtained from observing the climate prior to time t and can depend on how time t emissions might generate information about s for use in later periods.

These are clear advantages over a Monte Carlo analysis. We are now analyzing the conceptually correct problem for most interesting questions about uncertainty. However, there is a catch: the current problem can be much more difficult to solve. Solving the Monte Carlo problem merely requires solving a bunch of deterministic problems. It is not in principle any more difficult than solving the original deterministic problem. However, solving equa-

[6] Consider an example with $\tau = 1$. The optimal time 0 policy solves the first-order condition:

$$e_0^{unc} = \tilde{u}_0^{-1} \left(\beta \int_s \frac{\partial u_1(e_1^{unc}(T_1); T_1)}{\partial T_1} \frac{\partial T_1}{\partial e_1} \, p(s) \, \mathrm{d}s \right),$$

where $\tilde{u}_0 \coloneqq u_0'(\cdot)$. In contrast, the Monte Carlo approach yields

$$e_0^{MC} = \int_s \tilde{u}_0^{-1} \left(\beta \frac{\partial u_1(e_1^{det}(s_i); T_1)}{\partial T_1} \frac{\partial T_1}{\partial e_1} \right) p(s) \, \mathrm{d}s.$$

These two policies are equivalent if (a) $e_1^{unc}(T_1) = e_1^{det}(s_i)$ for each s_i and (b) $\tilde{u}_0^{-1}(\cdot)$ is linear. Condition (b) holds if and only if $u_0'''(\cdot) = 0$, so that the recursive model's agent would not undertake precautionary savings (which are never undertaken in a Monte Carlo analysis because the modeled agent is unaware of uncertainty). Condition (b) does not hold for most standard utility functions, including those used in integrated assessment models.

[7] Numerous textbooks derive the Bellman equation and show that its policy and value solutions match e^{unc} and W^{unc}. A standard reference is Stokey and Lucas (1989).

[8] If we used an infinite horizon setting ($\tau = \infty$) and did not allow per-period utility to vary with time, then we would have the special (and truly "recursive") case where $V_t(T_t) = V_{t+1}(T_t)$, so that we can drop the time index on the value function and write $V(T_t)$. We could also obtain that representation by tracking time in the state space.

tion (1) poses a very different challenge. If we knew $V_{t+1}(\cdot)$, the maximization is potentially straightforward (and almost surely simpler than maximizing over the τ-dimensional control vector in the deterministic problem): we could optimize policy at each time t separately from optimizing policy at any other time. However, we do not initially know $V_{t+1}(\cdot)$, as it depends on maximizations at time $t+1$ and beyond. The brunt of the computational challenge rests in approximating $V_{t+1}(\cdot)$ over the relevant state space, which is just T_t in our stylized example but is of much higher dimension in any standard climate-economy model. Judd (1998) and Miranda and Fackler (2002) provide textbook descriptions of the relevant computational methods, and recent reviews include Cai and Judd (2014), Maliar and Maliar (2014), and Fernández-Villaverde et al. (2016). We describe computational techniques in Section 6.

3 The Effect of Uncertainty on Policy

Most studies with recursive climate-economy models solve for each $V_t(\cdot)$ and then use the solution to simulate trajectories for policy variables and state variables. However, the value function itself contains valuable information about the underlying drivers of policy. Teasing apart and quantifying these underlying drivers makes the results of numerical studies of climate change both deeper and more generalizable. We here theoretically demonstrate these channels through which uncertainty affects policy. Only a few studies have previously used this type of information in their analysis, but we hope that this exposition will encourage future work to use the value function to learn about the underlying drivers of policy.[9] In the next section, we will quantify these drivers in an extension of the benchmark DICE integrated assessment model.

Again consider the setting described in Section 1, with ϵ_{t+1} now a stochastic shock that is independently and identically distributed over time. The value of ϵ_{t+1} will never be observed, but it will affect the observed temperature T_{t+1}. This shock in the temperature transition prevents the policymaker from learning the true value of s from a single observation of temperature. This stochastic shock represents the climate's natural variability.[10]

Assume that the policymaker learns about s as a Bayesian. At time t she observes the new realization of temperature and updates her prior distribution for s to form a posterior that she uses in equation (1). Let s and ϵ_t each be normally distributed and enter the temperature transition in a linear and separable fashion. In this case (which roughly matches nearly all of the literature to date), we have what is called a conjugate prior and thus know that the

[9] Lemoine and Traeger (2014) use the value function to tease apart the different channels through which potential tipping points affect policy. Lemoine and Traeger (2016a) quantify the channels through which aversion to ambiguity about a tipping point's threshold affects policy. Heutel et al. (2016) tease apart the channels through which tipping points and geoengineering controls interact.

[10] This shock does not represent imperfect monitoring of the true state of the climate because the shock's realized value affects the true value of the state and thus the policymaker's utility.

posterior distribution of s is also normal. We can write the parameters of that posterior as functions of the time t prior, of T_t, and of e_{t-1}.

Specifically, let the mean of the time t posterior for s be μ_t and the standard deviation be Σ_t. These parameters evolve according to the following closed-form equations:[11]

$$\mu_{t+1} = g(\mu_t, T_{t+1}, e_t, \epsilon_{t+1}) \text{ and } \Sigma_{t+1} = h(\Sigma_t, e_t).$$

We consider the standard case in which Σ_t declines deterministically.[12] The belief parameters μ_t and Σ_t do not directly affect the time t payoff, but they do affect the time t expectation operator. They are informational states that need to be tracked just like the more directly payoff-relevant state T_t. The Bellman equation (1) becomes:

$$V_t(T_t, \mu_t, \Sigma_t) = \max_{e_t} \left\{ u_t(e_t; T_t) + \beta E_t \left[V_{t+1}(T_{t+1}, \mu_{t+1}, \Sigma_{t+1}) \right] \right\}, \tag{2}$$

subject to the transition equations for T_{t+1}, μ_{t+1}, and Σ_{t+1}. The policymaker forms expectations of V_{t+1} by integrating over her subjective time t beliefs about s and over the objective distribution for the noise term ϵ_{t+1}. These expectations are conditioned on the current informational states μ_t and Σ_t as well as on T_t and on the policymaker's choice of e_t.

The policymaker's optimal choice of time t emissions is governed by the following first-order condition:

$$\frac{\partial u_t(e_t; T_t)}{\partial e_t} = -\beta E_t \left[\frac{\partial V_{t+1}}{\partial T_{t+1}} \frac{\partial T_{t+1}}{\partial e_t} + \frac{\partial V_{t+1}}{\partial \mu_{t+1}} \frac{\partial \mu_{t+1}}{\partial e_t} + \frac{\partial V_{t+1}}{\partial \Sigma_{t+1}} \frac{\partial \Sigma_{t+1}}{\partial e_t} \right].$$

Greater marginal utility from emissions (left-hand side) corresponds to less emissions and thus to a greater optimal tax on emissions. The right-hand side of this equation gives the welfare cost of additional emissions.[13] Pass the expectation operator through to decompose the right-hand side into its economic components:

$$\begin{aligned} \frac{\partial u_t(e_t; T_t)}{\partial e_t} = \beta \Bigg\{ & \underbrace{E_t \left[\frac{-\partial V_{t+1}}{\partial T_{t+1}} \right]}_{\text{A}} E_t \left[\frac{\partial T_{t+1}}{\partial e_t} \right] + \underbrace{Cov_t \left[\frac{-\partial V_{t+1}}{\partial T_{t+1}}, \frac{\partial T_{t+1}}{\partial e_t} \right]}_{\text{insurance}} \\ & + \underbrace{E_t \left[\frac{-\partial V_{t+1}}{\partial \mu_{t+1}} \frac{\partial \mu_{t+1}}{\partial e_t} \right] + E_t \left[\frac{-\partial V_{t+1}}{\partial \Sigma_{t+1}} \frac{\partial \Sigma_{t+1}}{\partial e_t} \right]}_{\text{active learning}} \Bigg\}. \end{aligned} \tag{3}$$

[11] For a specific example, see Kelly and Tan (2015). See also Cyert and DeGroot (1974) for more on conjugate priors.

[12] See Lemoine et al. (2016) for a setting in which the evolution of Σ_t is stochastic and potentially non-monotonic.

[13] If we explicitly wrote consumption separately from emissions, we could divide by the time t marginal utility of consumption to obtain the optimal time t tax on emissions.

We see four components that determine optimal time t emissions. We will soon see that the first term on the first line (labeled A) is itself composed of several components, including a precautionary abatement component and a passive learning component. The second component is insurance against the marginal effect of emissions on temperature covarying with the marginal cost of temperature. The last line determines how active learning (i.e., the ability to generate information about s through the choice of emissions) affects the optimal emission policy. We proceed to analyze each of these components in turn.

3.1 Certainty-equivalence, precautionary abatement, and passive learning

Begin with the component labeled A in equation (3). This component is the expected welfare cost of additional time $t+1$ temperature. It multiplies the expected increase in time $t+1$ temperature from additional time t emissions. A second-order Taylor expansion of $E_t[-\partial V_{t+1}/\partial T_{t+1}]$ around $z \coloneqq (E_t[T_{t+1}], \mu_t, \Sigma_{t+1})$ yields (noting that $E_t[\mu_{t+1}] = \mu_t$ for a Bayesian):

$$E_t\left[\frac{-\partial V_{t+1}}{\partial T_{t+1}}\right] \approx \underbrace{\left.\frac{-\partial V_{t+1}}{\partial T_{t+1}}\right|_{(E_t[T_{t+1}],\mu_t,0)}}_{\text{certainty-equivalent}} + \underbrace{\left\{\left.\frac{-\partial V_{t+1}}{\partial T_{t+1}}\right|_{z} - \left.\frac{-\partial V_{t+1}}{\partial T_{t+1}}\right|_{(E_t[T_{t+1}],\mu_t,0)}\right\}}_{\text{adjustment for future uncertainty}}$$
$$+ \underbrace{\left\{\frac{1}{2}\left.\frac{-\partial^3 V_{t+1}}{\partial T_{t+1}^3}\right|_{z} Var_t(T_{t+1}) + \left.\frac{-\partial^3 V_{t+1}}{\partial \mu_{t+1}\partial T_{t+1}^2}\right|_{z} Cov_t(T_{t+1}, \mu_{t+1})\right\}}_{\text{precautionary abatement}}$$
$$+ \underbrace{\frac{1}{2}\left.\frac{-\partial^3 V_{t+1}}{\partial \mu_{t+1}^2 \partial T_{t+1}}\right|_{z} Var_t(\mu_{t+1})}_{\text{signal smoothing}}. \tag{4}$$

The first term on the right-hand side is the certainty-equivalent tax on emissions: the welfare loss from a marginal increase in temperature is evaluated with s known to be fixed at its time t mean.[14] This tax would be the tax in a deterministic model that happened to reach T_t at time t with $s_i = \mu_t$. The term in braces on the first line adjusts the certainty-equivalent tax to use the correct continuation value. This adjustment changes the time t tax to reflect how uncertainty changes the marginal effect of temperature on future welfare. In other words, this adjustment accounts for how all of the components to be described below also affect policies and welfare after time t. This adjustment would be zero in the unrealistic case where the time t policymaker knew that s would be fixed at μ_t once time $t+1$ arrived,

[14] By fixing s at its time t mean rather than its time 0 mean, we are calculating the certainty-equivalent emission tax as the tax that would be optimal conditional on reaching the time t states and then ignoring uncertainty.

even though she is potentially unsure about the value of s that will govern the transition from time t to time $t+1$.

The second line on the right-hand side of equation (4) describes a precautionary abatement motive. In standard savings models, the third derivative of utility determines whether increasing uncertainty about future consumption leads an agent to save more, in which case we call the agent "prudent" (Leland, 1968; Drèze and Modigliani, 1972; Kimball, 1990). In our setting, reducing emissions (i.e., increasing "abatement") is a form of saving: it requires forgoing current utility in order to obtain additional future utility. Just as agents in standard settings undertake precautionary savings when the marginal utility of consumption is convex, here agents undertake precautionary abatement when the marginal welfare cost of temperature is convex (i.e., when $-\partial^3 V_{t+1}/\partial T_{t+1}^3 > 0$).[15,16] This precautionary abatement channel becomes stronger as the variance of temperature increases.

The covariance term in the second line of equation (4) accounts for how anticipated learning affects precautionary abatement. The sign of $-\partial^3 V_{t+1}/\partial \mu_{t+1} \partial T_{t+1}^2$ probably matches the sign of $-\partial^3 V_{t+1}/\partial T_{t+1}^3$ because the implications of raising μ_{t+1} are similar to the implications of raising T_{t+1}.[17] As shown in the transition equations for the full DICE model in the appendix, $Cov_t(T_{t+1}, \mu_{t+1}) > 0$ because high temperatures are a signal of high climate sensitivity: we tend to learn that future temperatures will drift up especially fast when we have already seen temperature start to rise. Anticipating learning about the climate's sensitivity to emissions then makes future temperature appear especially variable, because the policymaker does not know what she will learn but expects to learn something that will amplify whatever the observed near-term change in temperature turns out to be. The precautionary abatement motive is therefore strengthened by this positive covariance between the posterior mean of s and temperature.

The third line of equation (4) determines how passive learning (the exogenous arrival of information about s) affects the optimal emission tax. In particular, it describes how temperature affects the policymaker's ability to smooth welfare in response to whatever signal she receives about s. The curvature of V_{t+1} in μ_{t+1} captures the policymaker's ability to smooth the consequences of a higher estimate of climate sensitivity. Note that high s is

[15] Repeatedly applying the envelope theorem to the Bellman equation shows that $-\partial^3 V_{t+1}/\partial T_{t+1}^3$ is closely related to the third derivative of utility. Indeed, Sibley (1975) and Carroll and Kimball (1996) show that the value function can inherit the third derivative of utility.

[16] Another way to think about the sign of $-\partial^3 V_{t+1}/\partial T_{t+1}^3$ is to consider whether the policymaker would prefer to attach a mean-zero temperature risk to a state with more severe climate change or to a state with less severe climate change (cf. Eeckhoudt et al., 1995; Crainich et al., 2013). In the latter case, we have $-\partial^3 V_{t+1}/\partial T_{t+1}^3 > 0$.

[17] We could say that $-\partial^3 V_{t+1}/\partial \mu_{t+1} \partial T_{t+1}^2 \geq 0$ when the policymaker is "cross-prudent" in temperature with respect to expectations of climate sensitivity: the agent prefers to attach a mean-zero temperature risk to a state with lower expectations of climate sensitivity than to a state with higher expectations of climate sensitivity. However, note that we are here considering the derivatives of the value function rather than of per-period utility. For more on cross-prudence, see Gollier (2010).

bad: $-\partial V_{t+1}/\partial \mu_{t+1} > 0$. A positive value of $-\partial^2 V_{t+1}/\partial \mu_{t+1}^2$ indicates that the marginal cost of s increases in the level of s, as when damages are convex in the level of warming. The anticipated arrival of information about s makes μ_{t+1} variable from the perspective of time t: the policymaker expects to revise her beliefs when she sees the new information, but she does not know what that information will be. When $-\partial^2 V_{t+1}/\partial \mu_{t+1}^2 > (<)\ 0$, the variance $Var_t(\mu_{t+1})$ of future beliefs reduces (increases) the policymaker's expected future welfare. When $-\partial^3 V_{t+1}/\partial \mu_{t+1}^2 \partial T_{t+1} > 0$, higher temperatures hinder the policymaker's ability to smooth welfare in response to different signals about s.[18] In this plausible case, raising T_{t+1} (through higher e_t) increases the cost of the variance in μ_{t+1}. The policymaker then has an additional incentive to reduce emissions.[19]

3.2 Insurance

We now consider the other channels in equation (3). Begin with the insurance channel, which reduces optimal emissions if and only if $Cov_t\left[\frac{-\partial V_{t+1}}{\partial T_{t+1}},\ \frac{\partial T_{t+1}}{\partial e_t}\right]$ is positive. This channel depends on whether the marginal effect of emissions on temperature tends to be high when the marginal welfare cost of temperature is high or when the marginal welfare cost of temperature is low. In the former case, additional emissions have their strongest effect on the climate when climate change matters the most for welfare. The covariance is then positive, working to reduce optimal emissions and increase the optimal emission tax. In the latter case, additional emissions have their strongest effect on the climate when climate change matters the least for welfare. The covariance is then negative, working to increase optimal emissions and reduce the optimal emission tax.[20]

What sign should we expect this covariance to have? Continue to focus on uncertainty about the climate's response to emissions.[21] If we live in a world in which the climate is

[18] We could say that $-\partial^3 V_{t+1}/\partial \mu_{t+1}^2 \partial T_{t+1} \geq 0$ when the agent is "cross-prudent" in her expectations of climate sensitivity with respect to temperature: the agent prefers to attach a mean-zero shock to expected climate sensitivity (due, for instance, to a more precise signal) to a state with lower temperature than to a state with higher temperature. However, note that we are here considering the derivatives of the value function rather than of per-period utility. For more on cross-prudence, see Gollier (2010).

[19] This signal smoothing effect is related to the wealth effect described in Gollier et al. (2000) and Gollier (2001, Chapter 25). These analyses emphasize how increasing the informativeness of an anticipated signal induces precautionary saving but also reduces saving by increasing expected wealth, due to an enhanced ability to optimize in response to the signal. We see the precautionary effect in the $Cov_t(T_{t+1}, \mu_{t+1})$ term analyzed above.

[20] This logic is identical to the logic driving the consumption-based capital asset pricing model (Lucas, 1978; Breeden, 1979). There, we judge stocks to be risky based not on the variance of their returns but on the covariance of their returns with marginal utility. Stocks that pay off when marginal utility is high (i.e., when consumption is low) act as valuable hedges, which reduces the expected return that investors require to hold the stocks. Stocks that pay off when marginal utility is low (i.e., when consumption is high) make consumption more volatile, which increases the expected return that investors require to hold the stocks.

[21] See Gollier (2012), Litterman (2013), Weitzman (2013), Dietz et al. (2015), and Lemoine (2015) for

very responsive to emissions (s is large), then we might expect emissions to have a strong effect on temperature ($\partial T_{t+1}/\partial e_t$ to be large) and additional warming to be especially costly ($-\partial V_{t+1}/\partial T_{t+1}$ to be large). In this case, the covariance is positive and the insurance channel increases the value of emission reductions. This is the story analyzed in previous theoretical literature (Howarth, 2003; Sandsmark and Vennemo, 2007; Becker et al., 2010; Gollier, 2012; Dietz et al., 2015; Lemoine, 2015).[22]

3.3 Active learning

Finally, consider the terms on the bottom line of equation (3). These terms are nonzero only if emission decisions affect the mean and/or the variance of future beliefs.[23]

The first term on the bottom line of equation (3) accounts for how emissions change mean beliefs. Passing the expectation operator through, we have:

$$E_t\left[\frac{-\partial V_{t+1}}{\partial \mu_{t+1}}\frac{\partial \mu_{t+1}}{\partial e_t}\right] = Cov_t\left[\frac{-\partial V_{t+1}}{\partial \mu_{t+1}},\ \frac{\partial \mu_{t+1}}{\partial e_t}\right].$$

This expression recognizes that $E_t\left[\partial \mu_{t+1}/\partial e_t\right] = 0$, because a Bayesian policymaker cannot expect to learn in a particular direction or else she should have already updated her beliefs. If states in which additional emissions lead the policymaker to substantially raise her estimate of the climate's sensitivity to emissions ($\partial \mu_{t+1}/\partial e_t$ is large) correspond to states in which these larger estimates are especially costly ($-\partial V_{t+1}/\mu_{t+1}$ is large), then the covariance is positive and the policymaker reduces optimal emissions so as to avoid learning as rapidly. This story is in fact a plausible one: we would expect additional emissions to provide better signals when s is large, and because large s corresponds to large T_{t+1}, we might expect the welfare cost of higher mean estimates to be greater when s is larger. Intuitively, generating additional information through additional emissions increases risk: time $t+1$ beliefs become more variable from the perspective of time t. If the marginal cost of mean climate sensitivity ($-\partial V_{t+1}/\partial \mu_{t+1}$) is constant, then this additional risk does not reduce expected welfare, but

consideration of uncertainty about future consumption growth.

[22]The story could become more complicated once we realize that large s also implies that the climate has high inertia: as described in the scientific literature (e.g., Hansen et al., 1985; Raper et al., 2002; Baker and Roe, 2009; Urban and Keller, 2009), the climate system's response time falls when s increases because oceans at first absorb much of the additional heat from an increase in carbon dioxide. This increase in inertia would complicate our intuition about the sign of the covariance, because additional inertia should affect both derivatives in the covariance.

[23]In order for emissions to affect beliefs, emissions must interact with the uncertain parameter to determine the signal which the policymaker uses to update beliefs. If time $t+1$ temperature depends on emissions and climate sensitivity in a separable fashion, then the policymaker's ability to learn about climate sensitivity is not affected by decisions about emissions. In contrast, if carbon dioxide and climate sensitivity enter the temperature transition multiplicatively, then the marginal effect of s on temperature changes with the level of emissions. In this case, greater emissions can help the policymaker to back out the true value of s.

if the marginal cost of mean climate sensitivity is larger when the mean is higher, then this additional risk is undesirable. The policymaker then reduces emissions in order to avoid bearing this additional risk introduced by anticipated learning.

The second term on the bottom line of equation (3) captures how additional emissions generate information that allows the policymaker to develop more precise beliefs. Passing the expectation operator through and recognizing that the evolution of Σ_t is here deterministic, we have:

$$E_t\left[\frac{-\partial V_{t+1}}{\partial \Sigma_{t+1}}\frac{\partial \Sigma_{t+1}}{\partial e_t}\right] = E_t\left[\frac{-\partial V_{t+1}}{\partial \Sigma_{t+1}}\right]\frac{\partial \Sigma_{t+1}}{\partial e_t}.$$

The term $-\partial V_{t+1}/\partial \Sigma_{t+1}$ captures the value of information. It is positive because information has positive value. The term $\partial \Sigma_{t+1}/\partial e_t$ captures how emissions affect the variance of beliefs. It is negative when additional emissions allow for faster learning, as in many integrated assessment models. In that case, this active learning channel increases optimal emissions (and reduces the optimal tax on emissions) so as to generate additional information. This is the active learning channel that informal discussions in the literature have focused on.

This last channel is also the first channel that we have analyzed in which the most plausible consequence is to increase optimal emissions and thus reduce the optimal tax on emissions.[24] Therefore, when the climate's sensitivity to emissions is uncertain, we should expect this uncertainty to increase the optimal tax on emissions in the absence of learning; we should expect the introduction of passive learning (the anticipated and exogenous arrival of information) to further increase the optimal tax on emissions; and we should expect that the introduction of active learning (the endogenous generation of information) could reduce the optimal tax on emissions. We next quantify these channels in an extension of a benchmark integrated assessment model before reviewing the results of previous literature. Several of these channels depend on third derivatives and on cross-derivatives, so it is important to solve recursive climate-economy models with computational methods that do not a priori constrain these higher derivatives and cross-derivatives. We discuss computational approaches in Section 6.

4 Quantifying the Implications of Uncertainty for Policy

We now quantify the channels analyzed in Section 3 in a recursive extension of the benchmark DICE integrated assessment model of Nordhaus (2008). The DICE model couples a Ramsey-Cass-Koopmans growth model to calibrated modules that describe the transfer of carbon dioxide between the atmosphere and the ocean and that describe the evolution

[24]The appendix shows that using Epstein-Zin-Weil preferences introduces a new channel that reflects preferences over the temporal resolution of uncertainty. It explains why this channel is likely to increase the optimal emission tax in standard calibrations.

of atmospheric and oceanic temperature. Global warming reduces output directly. In each period, the policymaker chooses consumption, savings, and abatement to maximize intertemporal welfare. Savings increase future output by increasing the capital stock, and additional abatement increases future output by reducing future temperature. We extend DICE to include uncertainty about the climate's sensitivity to emissions. Following recent economic (Hwang et al., 2014; Kelly and Tan, 2015) and scientific (Roe and Baker, 2007; Roe, 2009) literature, the uncertain parameters are the feedbacks that determine the climate's sensitivity to emissions. This uncertainty can generate fat tails in the distribution of climate sensitivity. The appendix provides the complete model equations, and numerous other papers describe the DICE model in more detail. The appendix also describes how we map the analysis of Section 3 into this more complex numerical setting. Our model's code is available online.

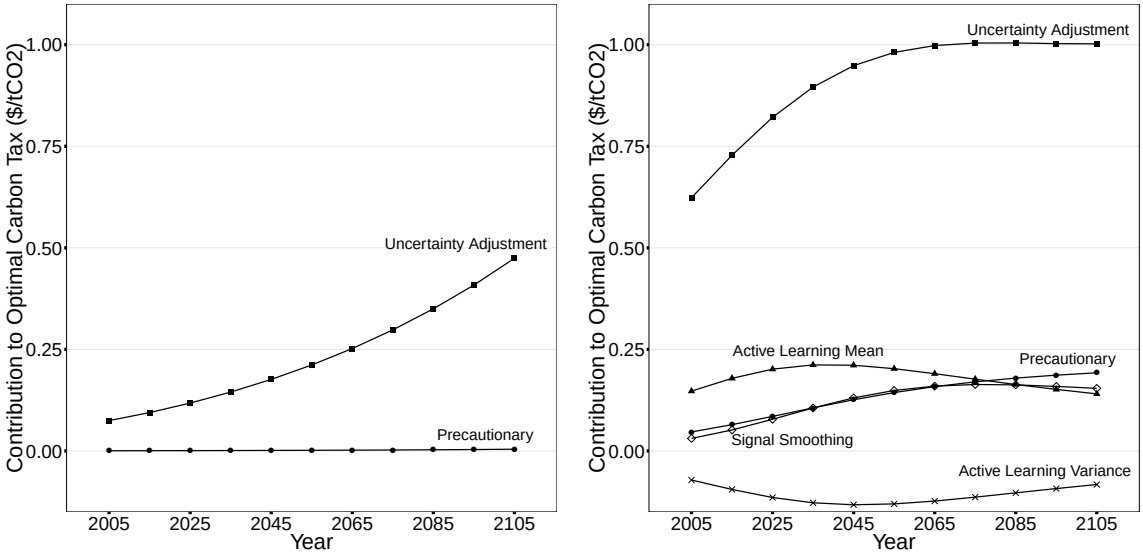
We find that uncertainty has only a small effect on the optimal emission tax. The optimal emission tax in the year 2005 would be \$7.80 per tCO_2 in the absence of uncertainty (plotted in the appendix), increasing very slightly to \$7.87 per tCO_2 in the presence of uncertainty, and increasing further to \$8.52 per tCO_2 when the policymaker anticipates learning about climate sensitivity. In the absence of uncertainty, the optimal tax would grow to \$23.25 per tCO_2 after fifty years. When climate sensitivity is uncertain (and just happens to take its mean value), the optimal tax grows to \$23.46 per tCO_2 after fifty years in the absence of learning and to \$24.57 per tCO_2 in the presence of learning.[25] Uncertainty does not have a significant effect on savings.

We are, however, here more interested in the channels through which uncertainty matters for policy than in the total magnitude of the effect. Figure 1 plots several of the channels analyzed in Section 3. It does not include the insurance channel because it is always zero: $\partial T_{t+1}/\partial e_t$ is not uncertain because of how the DICE model formulates its temperature transition (see appendix).[26] Both panels of the figure plot the evolution of each component when climate sensitivity takes its mean value (unbeknownst to the policymaker).

The left panel plots the components in the absence of learning. In this case, the mean belief μ_t does not evolve, so the signal smoothing and active learning channels are zero. As expected, the precautionary abatement channel does increase the optimal emission tax, but only by a very small amount. The variance of temperature at time $t+1$ is always fairly small from the perspective of time t, in part because the climate system's inertia means that uncertainty about climate sensitivity takes time to manifest as temperature. The important channel is the adjustment for future uncertainty. This channel is the dominant one because high or low values for climate sensitivity take time to manifest as warming. Raising emissions

[25]Kelly and Tan (2015) report that uncertainty has larger consequences for emission policy. Because they use a model that is structurally different from DICE, it is difficult to be sure where these differences stem from. Our results are similar to the small consequences of uncertainty in Hwang et al. (2014).

[26]Note, however, that $\partial T_{t+x}/\partial e_t$ is uncertain for $x > 1$, so the insurance channel is one of the components of the uncertainty-adjustment channel. We find that this channel is positive but very small when we consider $x = 2$. The small magnitude is likely due to the climate system's inertia.



(a) Without Learning (b) With Learning

Figure 1: The channels that determine how uncertainty about the climate's sensitivity to emissions changes the optimal emission tax, for settings without learning (left) and with anticipated learning (right). All simulations fix random variables at their mean values.

at any given time t has a strong effect on temperature only after several periods, which shows up by changing the continuation value V_{t+1}. The uncertainty adjustment grows stronger over time as the impacts of higher emissions become more imminent. In principle, one could decompose the uncertainty adjustment into contributions from insurance and precautionary abatement motives.

The right panel plots the components when the policymaker anticipates learning. Now the mean belief μ_t evolves with observations of temperature. The adjustment for future uncertainty is still the largest single component, again reflecting the delayed consequences of climate sensitivity for temperature. The precautionary abatement channel again increases the optimal emission tax. It is larger than it was in the absence of learning, largely because the covariance between temperature and mean beliefs is nontrivial. The precautionary abatement channel increases over the first half of the century. There are two competing effects governing how the precautionary abatement channel evolves over time. First, uncertainty about the climate sensitivity declines with learning, weakening precautionary motives. Second, warming over the first few decades raises the subjective variance of future temperature and strengthens the covariance between future temperature and the mean of our future beliefs, strengthening the precautionary abatement channel. The latter effect dominates the first effect over the coming century.

We also see three channels absent from the case without learning. First, the signal smoothing channel increases the optimal emission tax because the policymaker delays emissions in order to take advantage of new information about climate sensitivity. To our knowledge, this signal smoothing motive has not been discussed in previous work with recursive models. Second, the mean-belief component of the active learning channel increases the optimal emission tax because improving the informativeness of the temperature signal (through higher emissions) creates additional risk. Third, the belief-precision component of the active learning channel reduces the optimal emission tax because the policymaker can learn faster by increasing emissions.[27] Each component of the active learning channel strengthens over the first several decades as the variance of temperature and thus the variance of the mean of future beliefs each increase, before beginning to dissipate as the policymaker hones in on the true climate sensitivity. These signs for the signal smoothing and active learning channel's components were considered to be the most likely signs in the discussion in Section 3, and we see that the active learning channel always nets out to a positive value. Thus, we see that the introduction of passive learning works to raise the optimal emission tax and that making learning active then raises the optimal emission tax by a bit more.

5 The First Wave of Research with Recursive Models

We have thus far demonstrated how to use the value function to extract information about the channels through which uncertainty affects policy. We now review the main takeaways from previous work before discussing computational methods and suggesting directions for future work.

Figure 2 provides a timeline of published (black) and as-yet-unpublished (gray) work that uses recursive IAMs. A few observations emerge. First, Kelly and Kolstad (1999b) were far ahead of the rest of the literature. Only one other paper (Leach, 2007) was published using recursive techniques before 2013. Second, Christian Traeger's research led the resurgence of recursive IAMs. He published 5 papers (with coauthors) in this decade before anybody else had published one. Third, recursive IAMs are now hot. We count fifteen published papers and eight unpublished working papers since 2013, with only two published papers prior to 2013.

We now summarize what we see as the main lessons of this work. We focus on qualitative takeaways because model structures differ across papers and because these papers do not all

[27] In the DICE model, both parts of the active learning channel are implicitly captured by all the other channels when valuing the impact of time t emissions on the time $t+1$ expected continuation value. The appendix shows how the time $t+1$ variance and mean of the uncertain parameter are functions of time t temperature, which is a function of time $t-1$ emissions, so that the active learning channel is only explicitly present when looking two periods ahead. The optimal time t emission tax is the sum of the other plotted components (i.e., not including the active learning channel) and the certainty-equivalent tax. To assess the active learning channel, we plot $-\beta\, E_t\left[\frac{\partial V_{t+2}}{\partial \mu_{t+2}}\frac{\partial \mu_{t+2}}{\partial T_{t+1}}\frac{\partial T_{t+1}}{\partial e_t}\right]$ and $-\beta\, E_t\left[\frac{\partial V_{t+2}}{\partial \Sigma_{t+2}}\frac{\partial \Sigma_{t+2}}{\partial T_{t+1}}\frac{\partial T_{t+1}}{\partial e_t}\right]$.

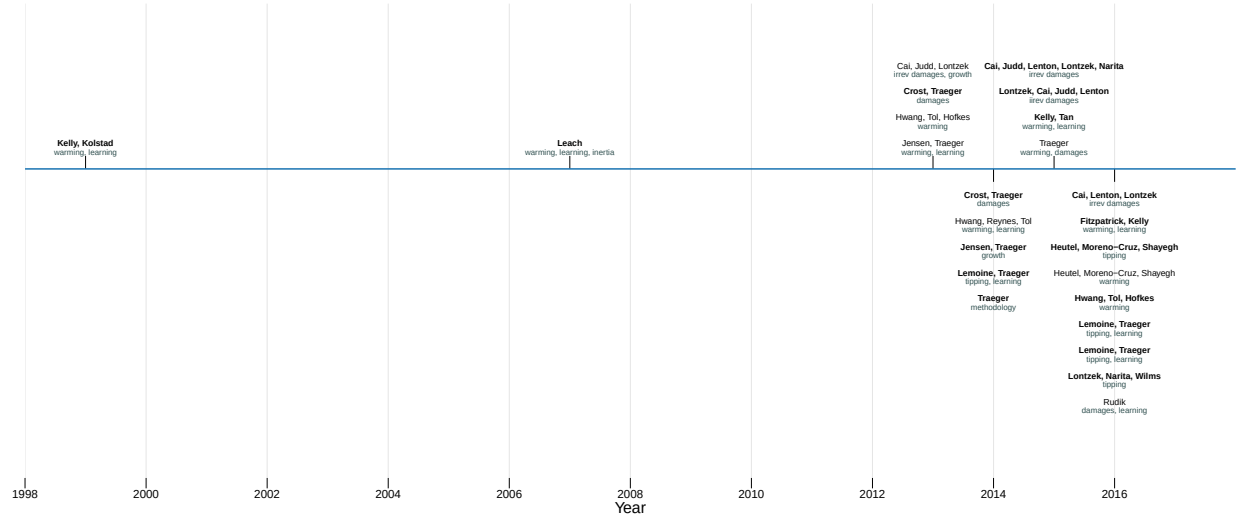


Figure 2: Timeline of publications (black) and as-yet-unpublished working papers (gray) that use recursive climate-economy integrated assessment models.

report comparable experiments.

1. Using common calibrations of recursive utility strongly reduces optimal emissions. Recursive utility, often implemented as Epstein-Zin-Weil (Epstein and Zin, 1989; Weil, 1989) preferences, separates risk aversion from the elasticity of intertemporal substitution, whereas the standard expected utility representation forces one to be the reciprocal of the other. Common calibrations set the elasticity of intertemporal substitution at a value strictly greater than 1 and set relative risk aversion to an even greater value. Much work has shown that increasing the elasticity of intertemporal substitution from standard expected utility calibrations strongly increases the optimal emission tax by reducing the consumption discount rate, and the high level of risk aversion (along with the implied preference for an early resolution of uncertainty) can also make policy especially sensitive to uncertainty (Cai et al., 2013; Crost and Traeger, 2014; Jensen and Traeger, 2014; Cai et al., 2016). The appendix extends the analysis of Section 3 to the case of recursive utility.

2. Other attitudes towards uncertainty matter less than does the adoption of recursive utility, probably because they do not affect the consumption discount rate as directly. In particular, a preference for robustness to alternative damage functions (Rudik, 2016) and aversion to ambiguity about a potential tipping point (Lemoine and Traeger, 2016a) each affect policy less than does the adoption of recursive utility.

3. In the expected utility framework, uncertainty about a scaling coefficient in climate damages and uncertainty about consumption growth each affect optimal policy less

strongly than does uncertainty about ultimate warming, which affects optimal policy less strongly than does uncertainty about how quickly damages increase with warming. Compare Crost and Traeger (2014), Jensen and Traeger (2014), and Kelly and Tan (2015).[28] All but one of these types of uncertainty increase the optimal emission tax. The exception is uncertainty about the damage function's scaling parameter.[29]

4. Fat-tailed uncertainty about future warming can reduce optimal emissions by a substantial amount, but the size of the effect is sensitive to our ability to learn about future warming (Hwang et al., 2014; Kelly and Tan, 2015). Uncertainty about warming also increases the cost of agreeing to inflexible limits on total warming rather than using new information to control policy optimally (Fitzpatrick and Kelly, 2016).

5. The policy implications of tipping points that affect the physical climate system vary strongly over plausible consequences of tipping and over assumptions about our ability to learn about tipping points, either prior to triggering them (Lemoine and Traeger, 2014) or after having triggering them (Lemoine and Traeger, 2016b). These policy implications also depend on whether deploying "geoengineering" technologies can mitigate the consequences of tipping (Heutel et al., 2016).

6. Allowing warming to raise the chance of irreversible reductions in economic output (Cai et al., 2013; Lontzek et al., 2015; Cai et al., 2016) or in environmental quality (Cai et al., 2015) can strongly reduce optimal emissions. This effect is especially strong under common calibrations of recursive utility (Cai et al., 2013, 2016).

6 Computational Discussion

The first wave of recursive IAMs explored the implications of different types of uncertainty for optimal emissions. We here describe the computational techniques used in this literature and point to improvements that could enable a second wave of research to ask questions that require more complicated models.

The standard solution technique used in the literature is value function iteration. Begin by supposing we have an infinite-horizon problem. Using the setting of Section 3, the Bellman

[28] But see our results in Section 4.

[29] Lemoine (2015) directly compares the implications of uncertainty about warming, uncertainty about the future variability of the weather, and uncertainty about future consumption growth for the social cost of carbon, rather than for the optimal emission tax. He shows that all increase the social cost of carbon, that uncertainty about consumption growth is the quantitatively most important of the three, and that precautionary abatement motives are more important than insurance motives. His work does not use a recursive IAM, which is why he does not analyze the optimal emission tax.

equation (1) becomes

$$V(T_t) = \max_{e_t} \left\{ u_t(e_t; T_t) + E_t\Big[\beta\, V(T_{t+1})\Big] \right\}, \quad \text{subject to } T_{t+1} = f(T_t, e_t, \epsilon_{t+1}; s). \tag{5}$$

A unique solution, $V(T_t)$, to equation (5) exists if u is real-valued, continuous, and bounded; $\beta \in (0, 1)$; and the feasible set of states for the next period is compact (Judd, 1998). Alternative sets of assumptions that guarantee a unique solution can be found in Stokey and Lucas (1989). We can ensure that the solution is approached in the limit as $j \to \infty$ by iterating on V as follows:[30]

$$V_{j+1}(x) = \max_{e_t} \left\{ u(e_t; T_t) + E_t\Big[\beta\, V_j(T_{t+1})\Big] \right\}.$$

If we have some arbitrary value function V_j, we can insert it into the right-hand side of the Bellman equation and undertake the maximization to recover a new value function V_{j+1}. If we repeat this process, we can eventually converge to the true value function for any initial V_0 (Judd, 1998). This is how the value function iteration algorithm works. In practice, however, this algorithm may not work perfectly due to numerical error in the maximization step or if the problem is ill-conditioned.

The first critical question is: how do we form a value function V_j? The conventional approach is to approximate V with a set of orthogonal basis functions $\phi(T)$:[31]

$$\tilde{V}_j(T) = \sum_{i=1}^{N} c_j^i\, \phi_j^i(T). \tag{6}$$

$c^i \in \mathbb{R}$ is the coefficient on the ith basis function. We solve for a new vector of c^i in each iteration j. To solve for N unknown scalars, we need at least N equations. Specifically, we will have N copies of equation (6), but evaluated at different points on a grid that we construct in our state space. We can write these N equations in matrix form as

$$\mathbf{V_j} = \mathbf{\Phi_j}\, \mathbf{c_j},$$

where $\mathbf{V_j}$ and $\mathbf{c_j}$ are $N \times 1$ and $\phi_\mathbf{j}$ is $N \times N$. The vector of coefficients is solved by a simple matrix inversion:

$$\mathbf{c_j} = \mathbf{\Phi_j^{-1}}\, \mathbf{V_j}.$$

In value function iteration, we assume some initial $\mathbf{c_0}$, maximize the right-hand side of the Bellman equation at N points in the state space, use the resulting vector of maximized

[30] Note that the subscript j here indexes the iteration, not time.

[31] The Stone-Weierstrauss theorem tells us that every continuous function on a closed interval can be approximated arbitrarily well by a polynomial.

values $\mathbf{V_1}$ to obtain a new vector of coefficients $\mathbf{c_1}$, and repeat until a predefined convergence criterion is satisfied.

Solving a finite-horizon problem works in a similar fashion. We begin with some terminal value function $\tilde{V}_\tau(T_\tau)$. If τ is sufficiently large and the policymaker discounts the future at some nonzero rate, then the choice of this value function will not matter for policy and welfare over some earlier, shorter horizon. The next step is to maximize the Bellman at time $\tau - 1$, which yields a vector of maximized values $\mathbf{V_{\tau-1}}$ that are used to recover the vector $\mathbf{c_{\tau-1}}$ and construct the time $\tau - 1$ value function approximant. We repeat the process, stepping back in time until time 0. At this point, we have a vector of coefficients (and thus a value function approximant) for every time period from 0 to τ.

The final question to answer is: how do we select our orthogonal basis functions and our set of points in the state space? The convention is to use Chebyshev polynomials for the basis functions and Chebyshev zeros for the points in the state space (after mapping the domain of the state space into the domain of the Chebyshev polynomials $[-1, 1]$), because of their favorable approximation properties when used together (Judd, 1998; Miranda and Fackler, 2002).[32] For each dimension d, we select a degree n_d Chebyshev polynomial and locate the n_d grid points at the n_d zeros of the polynomial. To construct the full polynomial and grid, we take a tensor product of the unidimensional polynomials and grid points. Other approximating functions used in the literature include neural networks (Kelly and Kolstad, 1999b, 2001), which can be universal approximators of continuous functions on compact sets, and splines (Fitzpatrick and Kelly, 2016), which have advantages if the value function is non-differentiable and which can target regions of the state space over which the value function is especially curved.[33,34]

[32] Boyd's Moral Principle (Boyd, 2000) essentially says to always use a Chebyshev basis, unless you are exceptionally sure that another basis is better for your problem (Fernández-Villaverde et al., 2016).

[33] Traeger (2014) compares cubic spline and Chebyshev bases. Cubic splines are the most common choice of spline basis. These ensure continuous first and second derivatives, and they restrict the third derivative to be piecewise constant. However, we have seen that the third derivative of the value function plays a critical role in determining policy under uncertainty. The choice of cubic splines may therefore not be innocuous.

[34] Some recent papers have adopted nonstandard approaches to solving the Bellman equation, such as using "logarithmic", state-separable basis functions (Hwang et al., 2013, 2014; Hwang, 2016) and fixing policy for some number of periods before approximating the remaining continuation value as a linear function of the per-period payoff (Heutel et al., 2015, 2016). We here caution against using less theoretically grounded methods for three reasons. First, several orders of value function derivatives and cross-derivatives matter for policy (see Section 3), and these alternative methods severely constrain these derivatives. Second, it is possible to "solve" a Bellman equation within a particular algorithm but for the solution to be incorrect. It is difficult to verify whether a solution to the approximated Bellman equation is correct, and theorems about function approximations provide extra confidence in the case of standard methods. Third, some of these methods may work in particular cases of a particular model, but they will almost surely not work in general cases. Yet the solutions can only be tested against known ones in particular (often deterministic) benchmark cases, even as a given paper is interested in the difference between a benchmark case and some other case. It is hard to know whether the cases of interest in a given paper are also particular cases in which the alternative algorithm happens to work and to know whether the reported differences in, for example,

6.1 Frontier Techniques

There are several drawbacks to the basic value function iteration approach outlined above. One is that it is typically slow, particularly if the discount factor β is near 1, as is common in models with an annual timestep. A second is that the computational complexity of the problem increases exponentially in the dimension of the state space (commonly referred to as the Curse of Dimensionality). In the interest of advancing the literature, we here describe some ways to reduce the computational cost and/or improve the accuracy of the standard tensor product approximation approach described above.[35]

Recall that we created our grid of points in the state space with a tensor product. Our grid has n^d elements if we have n unique points on each state and d states. Adding more states to the model rapidly increases the number of grid points and thus the number of required Bellman maximizations. This computational cost has limited IAMs. Researchers have circumvented this problem by finding clever ways to reduce the dimensionality of a complex climate-economy model. For example, Kelly and Kolstad (1999b) and Traeger (2014) employ models with one- and two-state climate systems instead of the five-state climate system of the DICE model (which is already a coarse approximation to full climate models).[36]

There are ways to reduce computation time without sacrificing model complexity. A simple way is to supply the solver used in the maximization step with gradient and Hessian information.[37] Most solvers allow the user to give analytic gradients and Hessians so that the solver no longer needs to approximate them, thereby improving both accuracy and speed. However, solving for these functions by hand, particularly in complex models like IAMs, can be troublesome and prone to human error. Autodifferentiation is a powerful technique that exploits the arithmetic structure of computer code to automatically generate gradients and Hessians. Since all programs are a combination of basic operations such as addition or exponentiation, we can use another pre-packaged autodifferentiation program to repeatedly apply the chain rule to arbitrary computer code in order to take derivatives of any program.

policy between these cases and the benchmark are due to the numerical methods.

[35] Many techniques common in other settings are often not helpful in solving integrated assessment models. First, Euler equation methods often do not help because IAMs contain more states than controls. Second, shape-preserving methods (Judd and Solnick, 1994; Cai and Judd, 2012) are often not justified because we do not know in advance that the value function is, for example, concave, and we may even know that it is not concave everywhere (due to nonlinearities in the forcing equation, for example). Third, policy iteration can be challenged by the likelihood of binding constraints and has not worked well in practice.

[36] These papers solve an infinite-horizon model with fewer states than the full model. Lemoine and Traeger (2014) approximate the omitted states as functions of the tracked states. Other papers have used finite-horizon methods that do not require iterating over the full grid of nodes as many times (e.g., Cai et al., 2012). Recent work has used sparse grids (discussed below) to solve a finite-horizon version of a full IAM (Rudik, 2016).

[37] Economists could also take greater advantage of high-performance computing facilities (Dongarra and van der Steen, 2012).

This can greatly reducing the potential for error in hand-coding these functions.

For infinite-horizon problems, we can accelerate convergence of the value function iteration algorithm outlined above by using a technique called "modified policy iteration". Modified policy iteration exploits two facts. First, maximizing the Bellman equation is the costly step in the algorithm. Second, the optimal policy function corresponding to the Bellman in iteration j will be very close to the optimal policy function for the Bellman in iteration $j+1$, especially on the grid of interest. Instead of maximizing the Bellman in each iteration to recover a new optimal policy, modified policy iteration periodically reuses the previous iteration's optimal policy to get the "maximized" Bellman value and solve for a new vector of coefficients for iteration $j+2$. Puterman and Shin (1978) demonstrate that this method will converge at least as fast as standard value function iteration.

A critical decision for speed and accuracy is the domain of the value function approximant. Selecting a domain that is wider than necessary will result in worse accuracy for a given grid, but selecting a domain that is too narrow can lead the solver to evaluate states that lie outside the domain of the approximant, where accuracy deteriorates. A general rule of thumb is to make the domain only as large as it needs to be to ensure that relevant areas of the state space do not transition outside the domain and to ensure that calculating expectations does not require placing too much weight on points outside the domain. If the domain is any larger, then we are fitting the value function in regions of the state space where we may never travel in simulations, which increases the computational burden with no benefit.

Some techniques adapt the domain of the value function approximant to the domain of simulated trajectories. The domain of the standard tensor product grid is shaped as a hypercube, but the domain of the simulated trajectories may not be. For example, they may be clustered along a diagonal of the grid in a three dimensional problem. In this case, large areas of the domain may never be reached in the simulations. Placing grid points in these regions adds computational cost with little benefit in improving the accuracy of our simulated outcomes. This tradeoff becomes especially important as the dimensionality of the problem increases.

Once we obtain the set of simulated trajectories from an existing solution to our model, perhaps from a version that uses a hypercube-shaped grid, we can adapt the domain of the value function approximant by using a change of coordinates. Judd et al. (2014b) and Maliar and Maliar (2015) propose a principal component transformation of the simulated data to rotate the coordinate system of the domain. When a hypercube is fit around the simulated points in the principal components system, the grid points are packed tightly into the important regions of the state space, allowing the modeler to obtain an approximation of the same accuracy but with fewer grid points.

In finite-horizon problems, a modeler can construct a time-adaptive grid. In the example of our simple IAM presented earlier, the only area of the state space we are concerned about at the initial time $t = 0$ is T_0. The domain of the value function at time $t = 0$ can be tightly bound around that value. Moving forward to time $t = 1$, the transition equations govern

the maximum and minimum possible temperature, which tells us how big the domain could possibly need to be conditional on the variance of the temperature shocks. In general, the domain becomes wider as we move forward in time to account for sequences of shocks that may reinforce each other.

Economists have recently begun to explore sparse grids (Smolyak, 1963; Krueger and Kubler, 2004; Judd et al., 2014b; Rudik, 2016). Sparse grids are constructed similarly to the standard tensor product grid; however, they omit certain subspaces of the full tensor product grid that are less important for approximation quality. The benefit of sparse grids is that the number of grid points only increases polynomially in the dimensionality of the problem. Although they use fewer grid points, sparse grids can be very accurate (Barthelmann et al., 2000). Indeed, one can think of sparse grids as the solution to an optimization problem: for an exogenously given number of nodes, find the approximation space that yields the greatest accuracy in terms of the L_2 and L_∞ norms, within the space of functions with bounded second-order mixed derivatives (Brumm and Scheidegger, 2016).[38] Sparse grids promise significant computational savings relative to standard methods.

7 The Second Wave of Research with Recursive Models

The first wave of research with recursive IAMs devoted substantial efforts to solving models with a single type of uncertainty. We have suggested that frontier techniques can open up a broader range of modeling possibilities. Recent work has begun applying such techniques to ask questions about model uncertainty (Rudik, 2016) and to calibrate the evolution of knowledge to the record of scientific progress (Lemoine et al., 2016). We here suggest some potentially helpful directions for a second wave of work with recursive IAMs, most of which depend on the ability to solve models with dramatically larger state spaces than used in the first wave of research.

1. Recursive IAMs should better model learning. Standard IAMs dramatically simplify the evolution of the climate and the economy, and recursive IAMs dramatically simplify the evolution of knowledge. This simplification is potentially critical because, as implied by Section 3, assuming that learning occurs too rapidly or that additional emissions generate too much information can qualitatively change policy conclusions. Actual knowledge about, for instance, future warming has been limited by the incompleteness of scientific observations, by the combination of complexity and coarseness in numerical climate models, and by the presence of multiple, potentially correlated uncertain parameters (e.g., climate sensitivity, ocean heat content, and aerosol forcing). Future work with recursive IAMs should seek not just to model the climate system but

[38] The bounded mixed derivative condition is met in most finite-horizon versions of the DICE model.

to model those features of the climate system that make climate science hard. Such work can answer important, but often overlooked, policy questions about the value of scientific monitoring and the optimal types of monitoring. As but one example, allowing for hidden states could be critical to asking questions about the value of an ocean monitoring system.

2. Economic uncertainties are more severe than scientific uncertainties, and are potentially as important for policy. Looking out 100 years, the overall quality of technology, the level of consumption, the quality of abatement technology, and preferences over environmental quality are at least as uncertain as any scientific aspect of the climate. Such uncertainties are very difficult to adequately calibrate or model, but doing so seems critical. Some of these uncertainties could be calibrated from asset prices, and others could be calibrated from historical time series.

3. Some of the most pressing unknowns relate to the possibility of catastrophes. In particular, many worry about the potential for abrupt, irreversible changes in the climate system ("tipping points") and about the potential for large declines in environmental quality or consumption in response to future warming. Thus far, work on physical tipping points has used reduced-form representations of the tipping process, and work on catastrophes has used either permanent shocks to output or reduced-form uncertainty about the curvature of the damage function. Future work should attempt a more structured approach to each problem, which may require extending IAMs to more faithfully represent the actual sources of information about these possible events as well as the actual, nonlinear mechanisms through which these events would unfold.

4. Future work should consider moving beyond the DICE model. To date, perhaps all recursive IAMs either have used variants of the DICE model (Nordhaus, 1992, 2008) or have used still simpler models. The DICE model has been an enormously valuable benchmark, and its transparent structure has been a boon to qualitative understanding. We showed above how to decompose the value function to glean theoretical insight from recursive settings. This type of approach to recursive modeling promises qualitative insight even from settings more complicated than DICE. Further, policymaker are in fact using IAMs for their quantitative policy conclusions (Greenstone et al., 2013). It is important to know how greater realism affects these numbers. Future advances in IAMs should consider extending benchmark settings to allow for features like multisector economies, endogenous or directed forms of growth, more realistic scientific modules, and a treatment of space. But this effort should be disciplined. The community should first aim to develop a set of stylized facts about asset prices, climate science, and economic activity that integrated assessment models should replicate.

We see the emerging potential to integrate these next-level considerations into full IAMs and to ask new questions that take the information structure of climate change as seriously as

the best-guess physical dynamics. We encourage recursive modelers to exercise creativity in moving towards these and other new directions.

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Appendices

The first appendix extends our theoretical analysis to the case of Epstein-Zin-Weil preferences. The second appendix describes our implementation of the DICE integrated assessment model and demonstrates some best practices.

Appendix A: Extension to Epstein-Zin-Weil Preferences

Epstein-Zin-Weil preferences (also called "recursive utility") allow for separation between preferences for smoothing consumption over risk and over time: they disentangle risk aversion from the elasticity of intertemporal substitution. Let ψ be the Arrow-Pratt measure of relative risk aversion and η be the reciprocal of the intertemporal elasticity of substitution. With Epstein-Zin-Weil preferences, the main text's Bellman equation becomes

$$V_t(T_t, \mu_t, \Sigma_t) = \max_{e_t} \left\{ \frac{c_t(e_t; T_t)^{1-\eta}}{1-\eta} + \frac{\beta}{1-\eta} \left(E\left[\left((1-\eta)\, V_{t+1}(T_{t+1}, \mu_{t+1}, \Sigma_{t+1}) \right)^{\frac{1-\psi}{1-\eta}} \right] \right)^{\frac{1-\eta}{1-\psi}} \right\},$$

for some consumption function $c_t(e_t; T_t)$. Expected utility corresponds to the special case of $\psi = \eta$. The first-order condition becomes

$$\begin{aligned} c_t^{-\eta} \frac{\partial c_t}{\partial e_t} = &- \beta \left(E_t \left[((1-\eta) V_{t+1})^{\frac{1-\psi}{1-\eta}} \right] \right)^{\frac{\psi-\eta}{1-\psi}} \\ &E_t \left[((1-\eta) V_{t+1})^{\frac{\eta-\psi}{1-\eta}} \left(\frac{\partial V_{t+1}}{\partial T_{t+1}} \frac{\partial T_{t+1}}{\partial e_t} + \frac{\partial V_{t+1}}{\partial \mu_{t+1}} \frac{\partial \mu_{t+1}}{\partial e_t} + \frac{\partial V_{t+1}}{\partial \Sigma_{t+1}} \frac{\partial \Sigma_{t+1}}{\partial e_t} \right) \right]. \end{aligned}$$

After a second-order Taylor expansion of the second term, the right-hand side becomes:

$$\begin{aligned} &- \beta \left(E_t \left[((1-\eta) V_{t+1})^{\frac{1-\psi}{1-\eta}} \right] \right)^{\frac{\psi-\eta}{1-\psi}} E_t \left[((1-\eta) V_{t+1})^{\frac{\eta-\psi}{1-\eta}} \right] \overbrace{E_t \left[\frac{\partial V_{t+1}}{\partial T_{t+1}} \frac{\partial T_{t+1}}{\partial e_t} + \frac{\partial V_{t+1}}{\partial \mu_{t+1}} \frac{\partial \mu_{t+1}}{\partial e_t} + \frac{\partial V_{t+1}}{\partial \Sigma_{t+1}} \frac{\partial \Sigma_{t+1}}{\partial e_t} \right]}^{\text{channels from main text}} \\ &\underbrace{+ \beta \left(E_t \left[((1-\eta) V_{t+1})^{\frac{1-\psi}{1-\eta}} \right] \right)^{\frac{\psi-\eta}{1-\psi}} Cov_t \left[((1-\eta) V_{t+1})^{\frac{\eta-\psi}{1-\eta}}, - \left(\frac{\partial V_{t+1}}{\partial T_{t+1}} \frac{\partial T_{t+1}}{\partial e_t} + \frac{\partial V_{t+1}}{\partial \mu_{t+1}} \frac{\partial \mu_{t+1}}{\partial e_t} + \frac{\partial V_{t+1}}{\partial \Sigma_{t+1}} \frac{\partial \Sigma_{t+1}}{\partial e_t} \right) \right.}_{\text{preference for temporal resolution}} \end{aligned}$$

The first line is as analyzed in the main text, with all channels adjusted by the factor

$$\left(E_t \left[((1-\eta) V_{t+1})^{\frac{1-\psi}{1-\eta}} \right] \right)^{\frac{\psi-\eta}{1-\psi}} E_t \left[((1-\eta) V_{t+1})^{\frac{\eta-\psi}{1-\eta}} \right].$$

Note that this factor is 1 in the case with $\psi = \eta$, which corresponds to the case of expected utility analyzed in the main text. The second line is novel. It is zero when $\psi = \eta$, which

explains why it is absent from the main text's analysis of a setting with expected utility. When $\psi > \eta$, the policymaker displays a preference for early resolution of uncertainty: she would pay money just to obtain information about a future risk sooner, even when she cannot act on that information. The covariance captures how the marginal value of emission reductions covaries with welfare. The second line increases the optimal emission tax if and only if this covariance is positive. When $\psi > \eta$ (as in standard calibrations), the covariance is positive when large V_{t+1} pairs with a small marginal value of emission reductions. For example, the covariance is positive when small values for climate sensitivity imply both high welfare and a small marginal value of emission reductions. We might in fact expect this combination to be the case, in which case the preference for temporal resolution channel increases the optimal emission tax. Intuitively, when the covariance is positive, reducing emissions smooths future welfare outcomes because it increases welfare most strongly in states with low welfare. Additional emission reductions thus help to resolve uncertainty about future welfare at an earlier date, and a policymaker with $\psi > \eta$ will pay for this earlier resolution of uncertainty.

Figure A-1 plots the quantitatively important channels in this setting with Epstein-Zin-Weil preferences. The left panels keep relative risk aversion at its DICE value of 2 but increase the elasticity of intertemporal substitution to 2/3, and the right panels keep the elasticity of intertemporal substitution at its DICE value of 1/2 but increase relative risk aversion to 3. These changes are in the same direction as changes suggested by some recent asset pricing models. The top panels plot the case of persistently uncertain climate sensitivity, and the bottom panels allow the policymaker to learn about climate sensitivity from observations of temperature (and to anticipate that she will do so). The active learning channel is constructed in the same way as in the main text's case of expected utility. Raising relative risk aversion does not affect the certainty-equivalent emission tax, but by reducing the consumption discount rate, raising the elasticity of intertemporal substitution increases the certainty-equivalent tax by over 40% (around \$3.25 per tCO_2) in the first period. In all cases, the temporal resolution channel is positive (as expected) but very small, in part because we do not change preferences by a large amount. The other channels are qualitatively similar to those in the main text's case of expected utility. We note three differences. First, by making the policymaker more sensitive to the risk induced by future revisions to beliefs, raising relative risk aversion increases the size of the signal smoothing channel in early years and leads it to decline monotonically over time (when random variables happen to take on their mean values). Second, raising relative risk aversion reduces the uncertainty adjustment by a bit. Third, raising relative risk aversion also makes the mean-belief component of the active learning channel start off negative, which makes active learning reduce the first period's optimal emission tax. These last two effects are a bit of a puzzle, but they could be related to how a preference for early resolution of uncertainty makes the policymaker dislike persistent shocks to mean beliefs, which are especially likely early on and, owing to interactions with inertia, when climate sensitivity is small.

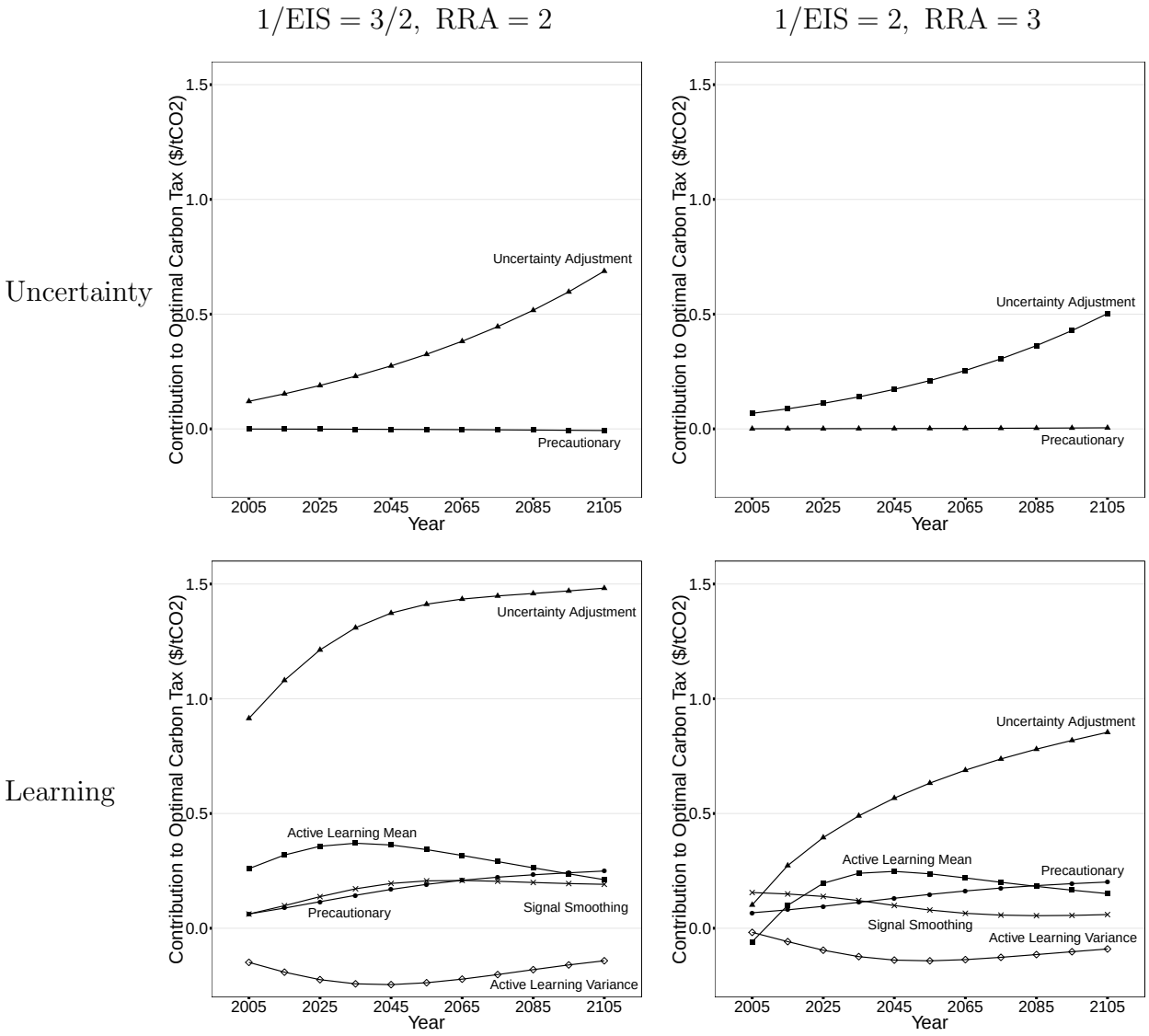

Figure A-1: The channels that determine how uncertainty about the climates sensitivity to emissions changes the optimal emission tax, for settings without learning (top row), with anticipated learning (bottom row), with less desire to smooth consumption over time (left column), and with greater aversion to risk (right column). All simulations fix random variables at their mean values.

Appendix B: The DICE Model

We now describe the model equations and calibration. We then extend our theoretical analysis of the channels through which uncertainty affects policy to the full numerical setting. We conclude with best practices for validating recursive models.

The DICE model is a Ramsey growth model coupled to a climate module. An infinitely lived representative agent aims to maximize the sum of the stream of discounted utility from consuming output. In decade t, the agent begins with some level of capital K_t. To produce gross output Y_t^g, the agent combines capital with labor L_t and technology A_t in a Cobb-Douglas production function:

$$Y_t^g = A_t \, L_t^{1-\kappa} \, K_t^{\kappa}.$$

Some of this output is lost to damages caused by surface warming T_t^s, so that output net of damages is given by

$$Y_t^n = \frac{Y_t^g}{1 + b_2 \, [T_t^s]^{b_3}}.$$

The agent can allocate her net output to consumption C_t or emissions abatement α_t, with residual output invested towards future capital, which depreciates at an annual rate δ_k. Per-period utility is:

$$u(C_t; L_t) = L_t \frac{(C_t/L_t)^{1-\eta}}{1-\eta},$$

with $\eta \geq 0, \neq 1$. Emissions e_t (net of abatement) are

$$e_t = 10 * \left[\sigma_t (1 - \alpha_t) Y_t^g + B_t\right],$$

where B_t gives exogenous emissions at time t and σ_t is the emission intensity of production at time t. We will constrain abatement to be less than the emissions generated by factor production. Emissions enter the atmospheric CO_2 stock M_t^{atm}. Between decades, CO_2 mixes between the atmospheric reservoir and reservoirs in the upper (M_t^{up}) and lower (M_t^{lo}) ocean.

Additional atmospheric CO_2 increases radiative forcing $F_t(M_t^{atm})$, which measures energy at the earth's surface. Forcing is given by

$$F_t(M_t^{atm}) = f_{2x} \, log_2(M_t^{atm}/M_{pre}) + EF_t,$$

where EF_t is exogenous forcing from other long-lived greenhouse gases and f_{2x} is the amount of forcing from doubling CO_2. Heat transfers between two reservoirs: one reservoir T_t^s that includes the earth's surface and the upper ocean, and a second reservoir T_t^o that reflects the lower ocean. Each reservoir's rate of warming is determined by three parameters: C_3 governs

the heat loss from the surface to the lower ocean, C_4 governs the heat loss from the lower ocean to the surface, and C_1 governs how quickly surface temperature responds to changes in forcing or to differences in the surface-ocean temperature gradient.

The climate sensitivity s determines how much the earth eventually warms after doubling of CO_2. We model the climate sensitivity as in Roe and Baker (2007):

$$s = \frac{\lambda_0}{1 - \Delta},$$

where λ_0 is the climate sensitivity for a reference system (lacking earth system feedbacks) and $\Delta < 1$ is the climate system's feedback factor.

To admit a closed-form updating rule for beliefs, we take Δ as the unknown parameter instead of s and, following Roe and Baker (2007), we assume that the agent believes $\Delta \sim \mathcal{N}(\mu_t, \Sigma_t)$ at time t. Her ability to learn about the true value of Δ is hindered by an independent and identically distributed temperature shock $\epsilon_t \sim \mathcal{N}(0, \sigma_T^2)$. Each decade, the agent updates the mean μ_t and variance Σ_t of her beliefs over Δ by using a normal-normal conjugate updating rule. The variance of her beliefs declines deterministically and monotonically over time.[39] However the mean of her belief at time $t+1$ is a function of the unknown temperature in time $t+1$, so μ_{t+1} is unknown at time t and evolves stochastically.

The model's exogenous economic processes are

$$\begin{aligned}
L_t &= L_0 + (L_\infty - L_0)\, g_{L,t} && \text{(Labor population)} \\
g_{L,t} &= \left[\exp(\delta_L\, t) - 1\right] / \exp\left(\delta_L\, t\right) && \text{(Labor population growth rate)} \\
A_t &= A_{t-1}/(1 - g_{A,t}) && \text{(Production technology)} \\
g_{A,t} &= 10\, g_{A,0} \exp(-\delta_A\, t) && \text{(Production technology growth rate)}
\end{aligned}$$

where index $t = 0$ indicates the year 2005, $t = 1$ indicates the year 2015, and so on. The model's exogenous climate-related processes are

$$\begin{aligned}
\sigma_t &= \sigma_{t-1}/(1 - g_{\sigma,t}) && \text{(Gross emissions per unit of output)} \\
g_{\sigma,t} &= g_{\sigma,0} \exp(-\delta_\sigma\, t) && \text{(Growth rate of gross emissions per unit of output)} \\
\psi_t &= \frac{a_0\, \sigma_t}{a_1 a_2} \left(a_1 - 1 + \exp(-g_\Psi\, t)\right) && \text{(Abatement cost coefficient)} \\
B_t &= B_0\, g_B^t && \text{(Non-industrial } CO_2 \text{ emissions)} \\
EF_t &= EF_0 + 0.1\, (EF_1 - EF_0) \min(t, 10) && \text{(Exogenous forcing)}
\end{aligned}$$

Table A-1 reports the values of the model parameters. The feedback mean is calibrated so the implied climate sensitivity of 3°C matches the DICE parameterization. We calibrate the

[39] The deterministic evolution of the variance is in a Markov sense. The agent knows the variance one decade ahead since it is entirely a function of time t variables, but she does not know it two decades ahead since T_{t+1} is unknown.

variance of the feedback term using the value in Roe and Baker (2007). The variance of the decadal temperature shock is obtained from Kelly and Kolstad (1999b).

Table A-1: The parameters of the dynamic stochastic DICE-2007 model.

Parameter	Value	Description
A_0	0.027	Initial production technology
$g_{A,0}$	0.009	Initial growth rate of production technology
δ_A	0.001	Change in growth rate of production technology
L_0	6514	Year 2005 population (millions)
L_∞	8600	Asymptotic population (millions)
δ_L	0.35	Rate of approach to asymptotic population level
σ_0	0.13	Initial emission intensity of output (Gigatons of carbon per unit output)
$g_{\sigma,0}$	-0.073	Initial growth rate of decarbonization
δ_σ	0.003	Change in growth rate of emissions intensity
a_0	1.17	Cost of backstop technology in 2005 ($1000 per ton of carbon)
a_1	2	Ratio of initial backstop technology cost to final backstop technology cost
a_2	2.8	Abatement cost function exponent
g_Ψ	0.05	Growth rate of backstop technology cost
B_0	1.1	Initial non-industrial CO_2 emissions (Gigatons of carbon)
g_B	0.9	Growth rate of non-industrial emissions
b_2	0.0028	Damage coefficient
b_3	2	Damage exponent
EF_0	-0.06	Year 2005 exogenous forcing (W/m^2)
EF_{100}	0.30	Year 2105 exogenous forcing (W/m^2)
κ	0.3	Capital elasticity in production
δ_κ	0.1	Annual capital depreciation rate
M_{pre}	596.4	Pre-industrial atmospheric CO_2 (Gigatons of carbon)
β	$1/1.015^{10}$	Discount factor
η	2	1/EIS, and RRA in the entangled preferences case
ϕ_{11}	0.811	Carbon transfer coefficient for atmosphere to atmosphere
ϕ_{12}	0.189	Carbon transfer coefficient for atmosphere to upper ocean
ϕ_{21}	0.097	Carbon transfer coefficient for upper ocean to atmosphere
ϕ_{22}	0.853	Carbon transfer coefficient for upper ocean to upper ocean
ϕ_{23}	0.050	Carbon transfer coefficient for upper ocean to lower ocean
ϕ_{32}	0.003	Carbon transfer coefficient for lower ocean to upper ocean
ϕ_{33}	0.997	Carbon transfer coefficient for lower ocean to lower ocean
C_1	0.22	Warming delay parameter
C_3	0.3	Parameter governing transfer of heat from ocean to surface
C_4	0.05	Parameter governing transfer of heat from surface to ocean
f_{2x}	3.8	Forcing from doubling of CO_2 (W/m^2)

Continued on next page

Table A-1 – continued from previous page

Parameter	Value	Description
σ_T^2	0.11	Temperature shock variance
K_0	137	Year 2005 capital (trillions of USD)
M_0^{atm}	808.9	Year 2005 atmospheric CO_2 (Gigatons of carbon)
M_0^{up}	1255	Year 2005 biosphere and upper ocean CO_2 (Gigatons of carbon)
M_0^{lo}	18365	Year 2005 lower ocean CO_2 (Gigatons of carbon)
T_0^{atm}	0.7307	Year 2005 atmospheric temperature (Degrees celsius)
T_0^{ocean}	0.0068	Year 2005 ocean temperature (Degrees celsius)
μ_0	0.6	Year 2005 feedback prior mean
Σ_0	0.13^2	Year 2005 feedback prior variance

We make two changes of variables to the model to reduce the computational burden. These changes do not alter the DICE model but only how it is represented in the computer code. In particular, we express both capital, K_t, and consumption, C_t, in terms of effective labor and technology:

$$k_t = \frac{K_t}{A_t^{1/(1-\kappa)} L_t},$$
$$c_t = \frac{C_t}{A_t^{1/(1-\kappa)} L_t}.$$

We keep utility in standard (not effective) terms so that the agent's utility function is

$$u(c_t; L_t, A_t) = L_t \, A_t^{(1-\eta)/(1-\kappa)} \frac{c_t^{1-\eta}}{1-\eta}.$$

The resulting problem yields the exact same solutions as the GAMS version of the DICE model, once modified to use a Markov representation of the forcing equation.

Our dynamic programming version of the DICE model is thus:

$$V_t(k_t, T_t^s, T_t^o, M_t^{atm}, M_t^{up}, M_t^{lo}, \mu_t, \Sigma_t) =$$
$$\max_{c_t, \alpha_t} \left\{ u(c_t; L_t, A_t) + \beta \, E\left[V_{t+1}(k_{t+1}, T_{t+1}^s, T_{t+1}^o, M_{t+1}^{atm}, M_{t+1}^{up}, M_{t+1}^{lo}, \mu_{t+1}, \Sigma_{t+1}) \right] \right\}$$

subject to transitions:

$$k_{t+1} = \frac{1}{A_{t+1}^{1/(1-\kappa)} \, L_{t+1}} \left[(1-\delta_k)^{10} \, A_t^{1/(1-\kappa)} \, L_t \, k_t + 10 \left((1 - \psi_t \alpha_t^{a_2}) Y_t^n - A_t^{1/(1-\kappa)} \, L_t \, c_t \right) \right],$$

$$\begin{bmatrix} M_{t+1}^{atm} \\ M_{t+1}^{up} \\ M_{t+1}^{lo} \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{21} & 0 \\ \phi_{12} & \phi_{22} & \phi_{32} \\ 0 & \phi_{23} & \phi_{33} \end{bmatrix} \begin{bmatrix} M_t^{atm} \\ M_t^{up} \\ M_t^{lo} \end{bmatrix} + \begin{bmatrix} e_t \\ 0 \\ 0 \end{bmatrix},$$

$$T_{t+1}^s = T_t^s + C_1 \left[F_{t+1}(M_{t+1}^{atm}) - f_{2x} \frac{1-\Delta}{\lambda_0} T_t^s + C_3 \, (T_t^o - T_t^s) \right] + \epsilon_t,$$

$$T_{t+1}^o = C_4 \, T_t^s + (1 - C_4) \, T_t^o,$$

$$\mu_{t+1} = \frac{\Sigma_t \, \gamma_t \, H_t + \sigma_T^2 \, \mu_t}{\Sigma_t \, \gamma_t^2 + \sigma_T^2},$$

$$\Sigma_{t+1} = \frac{\Sigma_t \sigma_T^2}{\Sigma_t \, \gamma_t^2 + \sigma_T^2},$$

where

$$\gamma_t = \frac{C_1 \, f_{2x} T_t^s}{\lambda_0},$$

$$H_t = T_{t+1}^s - \left(T_t^s + C_1 \left[F_{t+1}(M_{t+1}^{atm}) - f_{2x} \frac{1}{\lambda_0} T_t^s + C_3 \, (T_t^o - T_t^s) \right) \right].$$

Finally, we constrain industrial emissions to be nonnegative and we impose the resource constraint:

$$\alpha_t \leq 1,$$
$$A_t^{1/(1-\kappa)} \, L_t \, c_t + (\psi_t \alpha_t^{a_2}) \, Y_t^n \leq Y_t^n.$$

We take the double expectation over the feedback and temperature shock distributions by using Gauss-Hermite quadrature, with 7 unique quadrature points for the feedback distribution and 7 unique quadrature points for the temperature shock. This results in 49 total quadrature points covering the joint distribution. Our results are not sensitive to varying the number of quadrature points to either 25 or 81.

We solve the model by using value function iteration with a finite horizon. We set the horizon at 2555 and use a terminal value function where the policymaker has her initial 2005 beliefs about the uncertain parameter and all exogenously changing variables are held constant at their year 2555 values. In order to match the DICE model, we use a 10-year timestep,

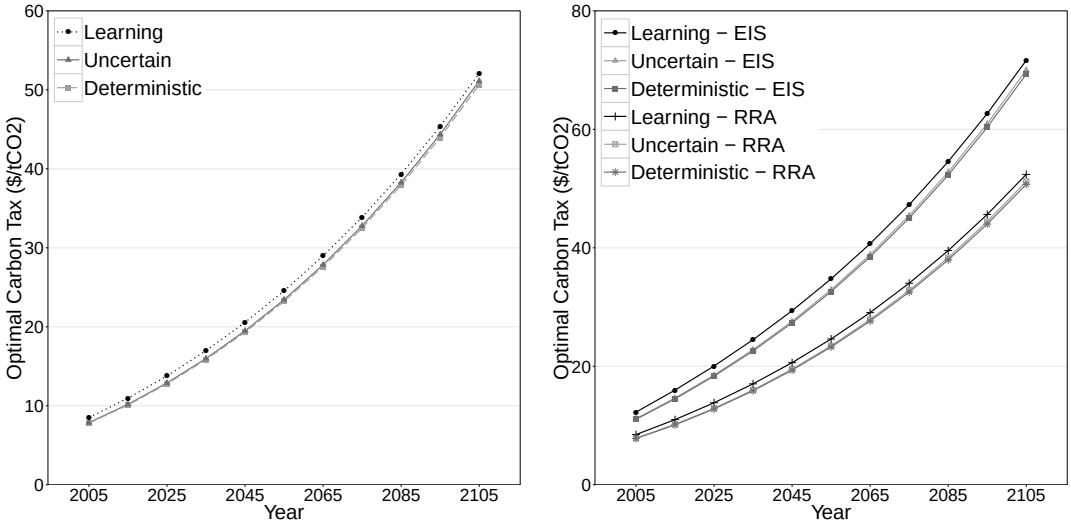

Figure A-2: The optimal emission tax with standard preferences (left) and with disentangled preferences (right). EIS indicates that the reciprocal of the elasticity of intertemporal substition has been lowered to 3/2, and RRA indicates that relative risk aversion has been raised to 3. Random variables happen to take on their mean values in every period.

but we recognize that an annual timestep could be important for a more thorough exploration of learning. Figure A-2 depicts the optimal emission tax trajectory in a deterministic version of our setting, when climate sensitivity is uncertain but the policymaker does not learn from observations, and when the policymaker anticipates learning from observations.

Finally, we extend the theoretical analysis of Section 3 to account for the additional states in DICE. The marginal cost of time t emissions is

$$-\beta E_t\left[\frac{\partial V_{t+1}}{\partial M^{atm}_{t+1}}\frac{\partial M^{atm}_{t+1}}{\partial e_t}+\frac{\partial V_{t+1}}{\partial T^{s}_{t+1}}\frac{\partial T^{s}_{t+1}}{\partial M^{atm}_{t+1}}\frac{\partial M^{atm}_{t+1}}{\partial e_t}\right].$$

Passing the expectation operator through, this becomes:

$$-\beta\Bigg\{\underbrace{E_t\left[\frac{\partial V_{t+1}}{\partial T^{s}_{t+1}}\right]}_{A}\frac{\partial T^{s}_{t+1}}{\partial M^{atm}_{t+1}}\frac{\partial M^{atm}_{t+1}}{\partial e_t}+\underbrace{E_t\left[\frac{\partial V_{t+1}}{\partial M^{atm}_{t+1}}\right]}_{B}\frac{\partial M^{atm}_{t+1}}{\partial e_t}\Bigg\}.$$

Term A is essentially as analyzed in the main text. Term B can be analyzed in a directly analogous fashion. For the main text's figure, we combine the channels in term A with the corresponding channels in term B. We lose the insurance term because $\partial T^{s}_{t+1}/\partial M^{atm}_{t+1}$ is not uncertain from the perspective of time t: the uncertain parameter Δ does not interact with forcing in the temperature transition. We also lose the active learning channel because neither the transition equation for μ_{t+1} nor the transition equation for Σ_{t+1} depends on time

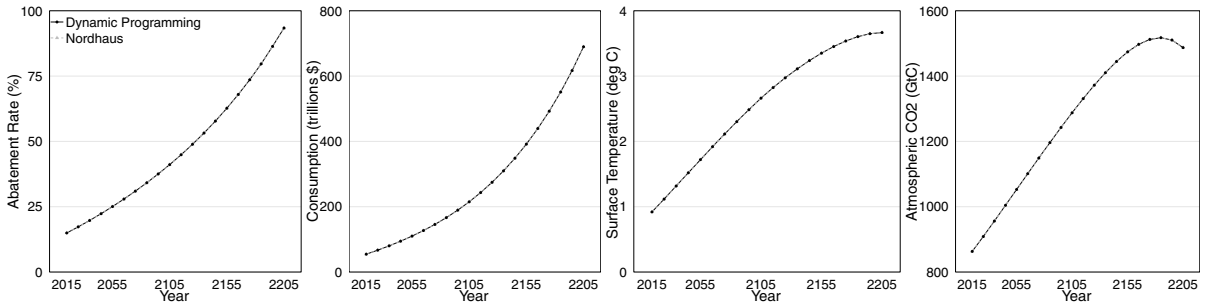

Figure A-3: Optimal trajectories for the GAMS DICE-2007 model (triangles) and our dynamic programming version of DICE-2007 (circles) under the standard DICE parameterization.

t emissions, once we substitute for T^s_{t+1} in H_t. The active learning channel and the insurance channel both reappear after a longer interval lapses, so that they help to determine the adjustment for future uncertainty.

The second-order Taylor expansions of terms A and B appear to be adequate. The maximum relative error between the sum of our Taylor approximation terms and the actual optimal carbon tax is 2×10^{-4} for the learning results in the main text. The average relative error is 8×10^{-5}. The maximum and average relative errors for the uncertainty results are 2×10^{-6} and 2×10^{-7}.

Best Practices

Quantifying errors in solutions and providing sufficient information for replication is critical. Here we demonstrate some best practices using the dynamic stochastic version of DICE developed for this paper. We first demonstrate the accuracy of our model. There are several ways to test model accuracy, but we begin by comparing optimal trajectories from deterministic runs of our dynamic programming model to the deterministic DICE-2007 model solved in GAMS.[40] In order to make an apples to apples comparison, we change the GAMS model to make it Markov: we rewrite the forcing equation so that current forcing is a function of current atmospheric CO_2 and not the average of current and next period's atmospheric CO_2.[41]

[40] Cai et al. (2013) demonstrate accuracy of their dynamic programming solution by comparing deterministic runs of their dynamic programming model to solutions of their model solved using more accurate optimal control techniques.

[41] Replicating known results is not always a good check. Many recursive IAMs use an annual timestep, whereas the DICE model uses a ten-year timestep. Optimal policies should vary between these settings. Further, finding only small errors in simulated trajectories in one parameterization of the model (such as the standard DICE parameterization) does not guarantee that simulations traveling to other areas of the state space will be as accurate.

Table A-2: Relative errors between GAMS solution and the dynamic programming solution to the DICE-2007 model.

	Abatement Rate	Consumption	Temperature	CO_2
Maximum Relative Error	3.9×10^{-3}	7.2×10^{-4}	3.6×10^{-4}	3.0×10^{-4}
Average Relative Error	1.1×10^{-3}	2.8×10^{-4}	2.2×10^{-4}	1.9×10^{-4}

Figure A-3 displays the optimal trajectories for the two models from 2005 to 2205, focusing on the abatement rate control, the consumption control, the surface temperature state, and the atmospheric CO_2 state. In these plots, we solve the dynamic programming model on the larger domain that we use when the climate sensitivity is uncertain, and we set $\mu = 3$ for the Smolyak algorithm.[42] Solving the problem on a tighter domain, or omitting the unnecessary belief states for this setting, would indeed lead to greater accuracy but would not provide a fair assessment of the accuracy of the solutions used to analyze uncertainty about climate sensitivity.[43]

All plotted state and control trajectories are extremely close, and visually indistinguishable. Table A-2 displays the maximum and average relative errors between the GAMS and dynamic programming trajectories. The abatement rate displays the greatest error out of all the trajectories but is still very accurate and diverges from the GAMS solution by one-tenth of a percent on average. The maximum and average errors for the other trajectories are extremely small and only a few hundredths of a percent.

In addition to checking whether the model has acceptable error in optimal trajectories, we can check for internal consistency in the model.[44] In many macroeconomic models this is done by analyzing the Euler equation residuals. Santos (2000) notes that under certain conditions the size of the Euler residuals is the same order of magnitude as the approximation error in the policy function. However, the DICE model does not satisfy these conditions due to the concavity of forcing and to its use, in some cases, of a finite horizon. Instead we can study residuals of other equations that must be satisfied along an optimal trajectory. These do not have a direct mapping into errors in policy or welfare, but nonetheless can give us some insight into the accuracy of our approximation. Judd et al. (2014a) and Fernández-Villaverde et al. (2016) give some suggestions regarding residuals to test, such as Bellman errors, first-order condition errors, or a χ^2 accuracy test.

[42] The model solves in 15-20 minutes on a 28 core machine. $\mu = 3$ implies 9 unique collocation points for each dimension, although there is not 9^8 points on the resulting grid since it is not constructed from a full tensor product.

[43] Another way to validate the model and code accuracy is to simulate our model forward using the optimal trajectories from the Markov version of the GAMS DICE-2007 model. Doing this, we find that the state trajectories are nearly identical.

[44] One should always examine the value function derivatives to ensure that they are sensible, and one should always ensure that the solution is not particularly sensitive to the domain or degree of approximation.

Table A-3: First-order condition errors of the learning model in log10 units.

	Abatement Rate	Consumption
Maximum Error	-4.75	-6.70
Average Error	-5.84	-7.31

Here we demonstrate testing first-order condition errors. The first-order conditions determine optimal policy, so ensuring that this error is small along simulated trajectories is critical for getting the correct policy implications of changes in risk and uncertainty. Following the convention for residual analysis, we calculate first-order condition errors by rearranging the first-order conditions so that all terms are on the left-hand side of the equation and 1 is on the other side, subtracting 1 from the left-hand side, and taking the base 10 logarithm of the absolute value. If there were no approximation error, then taking the base 10 logarithm would produce negative infinity, but this does not happen in practice due to errors in the approximation of the value function and due to truncation and rounding during simulation. Table A-3 reports the error in the first-order conditions of our learning model's simulated trajectory in log10 units. Numbers that are larger in magnitude imply smaller errors in the first-order conditions. Along the optimal trajectories, the errors in the first-order conditions are small. The average abatement error is -5.84, and all are smaller than -4.75. Consumption errors are even smaller (-7.31) on average, and all consumption errors are less than -6.70.

Finally, we report the domain of approximation in Table A-4. The domains for both temperature states are matched since a tighter ocean state domain will always result in the ocean state transitioning outside the domain because next periods ocean temperature is a convex combination of current surface and ocean temperature. We select the domain for the two ocean CO_2 states by using the transition equations to infer the tightest bounds that would keep the next period's ocean CO_2 states within the domain.[45] Results do not change if we select a tighter domain for the ocean CO_2 states and allow the continuation value to be evaluated outside the domain during the approximation step. The lower bound for the atmospheric CO_2 state was selected so that, conditional on how we construct the domain for the ocean CO_2 states, the upper ocean CO_2 state does not immediately jump outside the domain during simulation. The variance of beliefs is naturally bounded between the initial belief and zero. The effective capital domain is selected based on the tightest domain that would allow for convergence of the learning model. We select the domain for the mean of the feedback distribution to be accurate under the test against the GAMS version of DICE and to take advantage of the fact that we only simulate the model with all random variables fixed at their mean values. Note that, during the value function approximation steps, the continuation value is evaluated outside the domain at a number of the quadrature points

[45] Due to numerical error, some grid points do transition outside the domain, but only by a very small amount.

Table A-4: Domain bounds for the approximation space.

	k_t	T_t^s	T_t^o	M_t^{atm}	M_t^{up}	M_t^{lo}	μ_t	Σ_t
Lower Bound	1.7	0	0	580	1229	18204	0.4	0
Upper Bound	6	10.6	10.6	1700	2310	47064	0.8	0.13^2

over next period's feedback mean. Since our results are not sensitive to the choice of the number of quadrature points (and thus are not sensitive to the number of mean quadrature points that are evaluated outside or inside the domain during approximation), we take our results to be sufficiently accurate. A wider domain will be required if a user wants to explore simulations with noise shocks or with a true value for the feedback parameter that is different than 0.6.