Economic Models
and Mean-field Games Theory

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Economic Models and Mean-field Games Theory

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1

Introduction

Mathematical methods are central in modern economic theory as they allow for testable models in a wide range of complex problems. A further advantage of the mathematical formalism is that it enables authors to formulate and solve models in a unified language. While not all economists agree that the behavior of agents can be reduced to a precise mathematical formulation, utility maximization principles and game-theoretical equilibria explain, at least partially, many economic phenomena.

The concept of competing agents is illustrated in the works of A. Cournot and L. Walras. Indeed, Walras, also known as the founder of the *École de Lausanne*, refers to Cournot in 1873, as the first author to seriously recur to the mathematical formalism in investigating economic problems. Cournot duopoly model is one of the earliest formulations of a non-cooperative game. This pioneering work sets up the foundation of contemporary game theory. Walras developed a first theory of a competitive market (general) equilibrium. W. S. Jevons and C. Menger are also well known for their influence on the presence of mathematical formalism in economics.

The mathematical formulation of economic problems has attracted the attention of notable mathematicians, including E. Borel and J. von Neumann. In the mid-20th century, in the paper [131], J. Nash developed a concept of equilibrium that is fundamental in modern game theory. The arguments in that paper rely on a fixed-point theorem, due to S, Kakutani. For his "contributions to the analysis of equilibria in the theory of non-cooperative games", J. Nash (together with C. Harsanyi and R. Selten) was awarded the Nobel Prize in Economics in 1994.

Another Nobel Prize Laureate, R. Aumann, introduced in 1964 the idea of an economy with a continuum of players, which are atomized in nature [14]. In that paper, Aumann argues that only for an economy with infinitely many participants it is reasonable to assume that the actions of individual agents are negligible in determining the overall outcome.

In 1995, the Nobel Prize was awarded to R. Lucas, for the development and applications of the hypothesis of rational expectations, in the early 70's, [144]. This hypothesis states that economic agents' predictions of economically relevant quantities are not systematically wrong. More precisely, the subjective probabilities as perceived by the agents agree with the empirical probabilities. After the introduction of this framework, an important paradigm in economic theory is based on three hypotheses: efficient markets, rational expectations, and representative agent.

It is only around the 90's that alternatives to the representative agent model begin to be considered in mainstream economics. The idea of heterogeneous agents, as suggested in the works of S. Aiyagari [10], T. Bewley [27], M. Huggett [107] and P. Krussel and A. Smith [115], points out in an alternative direction. In this formulation, agents in the economy are characterized by different levels of the model's variables. For example, individuals can have distinct income or wealth levels.

In the theory of mean-field games (MFG), the concept of Nash equilibrium and the rational expectation hypothesis are combined to produce mathematical models for large systems, with infinitely many indistinguishable rational players. The term *indistinguishable* refers to a setting where agents share common structures of the model, though they are allowed to have heterogeneous states. In other terms, the MFG theory enables us to investigate the solution concept of Nash equilibrium, for a large population of heterogeneous agents, under the hypothesis of rational expectations.

1.1 Mean-field games

The mean-field game formalism was developed in a series of seminal papers by J.-M. Lasry and P.-L. Lions [118, 119, 120] and M. Huang, R. Malhamé and P. Caines [103, 106]. It comprises methods and techniques to study differential games with a large population of rational players. These agents have preferences not only about their state (e.g., wealth, capital) but also on the distribution of the remaining individuals in the population. Mean-field games theory studies generalized Nash equilibria for these systems. Typically, these models are formulated in terms of partial differential equations, namely a transport or Fokker-Plank equation for the distribution of the agents coupled with a Hamilton-Jacobi equation.

From the beginning, applications of MFG ideas and methods to problems arising in Economics and sustainable development were investigated by various authors; see, for instance, the work of O. Guéant [93], A. Lachapelle, J. Solomon and J. Turinici [116] and J.-M. Lasry, P.-L. Lions and O. Guéant [121, 122]. Regarding the Economics community, some of the ideas in the paper by P. Krussel and A. Smith [115] closely resemble the intuition made rigorous by some classes of mean-field game models. From a purely game-theoretical perspective, similar ideas were considered in [109]. The importance of MFG models stems from the fact that it allows a systematic formalization of two workhorses of the modern economic theory, namely, the framework of rational expectations and heterogeneous agent models. In several instances of economic interest, ideas and methods resembling those formalized by the MFG theory have been addressed. This is the case of [108], [152], [129], [99], [12] and [2], to name just a few.

Recently, substantial progress has been achieved in the study of the connection between MFG models and problems in Economics and Finance. See, for instance, the work of R. Lucas and B. Moll [145] and the developments in the work of Y. Achdou, F. Buera, J.-M. Lasry, P.-L. Lions and B. Moll [4] and Y. Achdou, J. Han, J.-M. Lasry, P.-L. Lions and B. Moll [7]. This approach has been fruitful in the study of economic models, while pointing out original directions of research in Mathematics. The potential applications of these techniques are various; these include new technology adoption, economic policies regarding income and wealth inequalities and the sustainable management of non-renewable resources.

In addition to applications in mathematical economics, mean-field game models arise in diverse areas ranging form crowd and population dynamics [32] to nonlinear estimation [136], and machine learning [137].

An important research direction in the theory of MFG concerns the study of the existence and regularity of solutions. Well-posedness in the class of smooth solutions was investigated, both in the stationary and in the time-dependent setting. Time-dependent, second-order problems, are considered in [45], for purely quadratic Hamiltonians, as in [46]. The general case was systematically investigated in [139, 82, 81], for sub and superquadratic Hamiltonians. The case of logarithmic nonlinearities was addressed in [78]. Additional results on the regularity of solutions are reported in [79, 80]. The stationary problem was considered in [85, 83, 77, 140] (see also [71]). Obstacle type problems and weakly coupled systems were examined in [76] and [75]. Congestion problems were considered in [72], in the stationary case, and [90] for time-dependent problems. In [92], the existence of weak solutions for the congestion problem was proved (see also the approach with density constraints in [147] and [128]). Finally, logistic-type dynamics was investigated in [84]. Weak solutions were addressed originally in [118] (stationary case) and [119, 120] (time dependent problems). Then, they were systematically investigated in [141, 142, 44, 43]. Many of these results concern simplified problems (e.g., periodic setting) and do not extend to the models we discuss here. A novel approach to construct weak solutions for MFG using monotonicity methods is presented in [63] (see [11] and [86] for applications to numerical methods using mononocity techniques).

A notable effort was made to study these problems numerically. Computational methods are particularly relevant in economic models as they rarely admit explicit solutions. Recent advances in this field are reviewed in [3], in the papers [5, 6, 8, 47, 41, 9, 20], and in [98, 97]. Monotonicity techniques are used to develop numerical methods for MFG in [11] and [86]. See [4, 7, 116] for economically motivated problems. More recently, numerical results for mean-field games on networks are reported in [40].

Finite state mean-field games have also been considered in the literature but will not be discussed here. Instead, we refer the reader to [73, 74, 70, 95, 94, 64] for the general theory, the reference [26] for applications to socio-economic sciences (also see [157]), as well as to [88, 89, 86] for numerical methods.

Applications with mixed populations or with a major player were examined in [101, 102, 133]. Linear quadratic problems have been considered from diverse perspectives, see [103], [104], [16], [105], [96], [132], [19], [24], [23], [123], [54], [155], and [62]. In the context of MFG, crowd and population dynamics were investigated in [56, 31, 32, 33]. The consensus problem was studied in [134].

The rigorous derivation of mean-field models was considered in particular cases in the original papers by Lions and Lasry. Further developments, using the theory of nonlinear Markov processes were obtained in [114], [111], and [112] (see also the monograph [113]), and using PDE methods in [18, 62]. For finite state problems, the N-player problem was studied in [74] where a convergence result was established. For earlier works in the context of statistical physics and interacting particle systems, see [150]. Recent results in this field include [65] and [117].

A related problem is mean-field control. For that, we refer the reader to the monograph [21], as well as the papers [50, 134], comparing these different approaches.

1.2 Mean-field games and economic theory

In traditional economic models, the simplifying assumption that all agents are identical (representative agent assumption) is often adopted. In contrast, heterogeneous agent problems allow the study of questions in which the differences among agents are of primary relevance. Matters such as income inequality or wealth distribution are inherently associated with differences among agents. In other problems, the effect of heterogeneity may not be adequately captured by the representative agent assumption.

Mean-field games model large populations of rational, heterogeneous agents. The analytical spectrum of the MFG framework accommodates preference structures and effects that depend on the whole distribution of the population. A rational agent is an agent with defined preferences that he or she seeks to optimize. In the vast majority of cases, these preferences can be modeled through a utility functional. Rationality means that the agent always acts optimally seeking to maximize its utility. Finally, MFG are closely linked to the assumption of rational expectations. The forecast of future quantities by the agents is an essential part of any economic model. The rational expectation hypothesis states that predictions by the agents of the value of relevant variables do not differ systematically from equilibrium conditions. This hypothesis has several advantages: firstly, it can be a good approximation to reality, as agents who act in a non-rational way will be driven out in a competitive market; secondly, it produces well-defined and relatively tractable mathematical problems; finally, because of this, it is possible to make quantitative and qualitative predictions that can be compared to real data. In mean-field game models, the actions of the agents are determined by looking at objective functionals involving expectated values with respect to probability measures that are consistent with the equilibrium behavior of the model. This contrasts with the adaptive expectations approach, where the model for future behavior of the agents is built on their past actions.

Agent-based computational methods are very popular tools to study heterogeneous agent economic problems. Unfortunately, numerical methods do not provide analytical models from which qualitative properties can be derived. In modern macroeconomics, an important role is played by dynamic stochastic general equilibrium models. These aim to understand the microfoundations of macroeconomics. MFG methods are equilibrium models where all agents are rational. In simple problems, fundamental questions such as uniqueness, existence or stability were investigated extensively. However, many MFG problems arising in mathematical economics raise issues that cannot be dealt with the current results.

The reader interested in the connection between mean-field games and economics is referred to the work of J.-M. Lasry, P.-L. Lions and O. Guéant [121, 122]. Developments in the computation of MFG equilibrium arising in economic problems are reported by A. Lachapelle, J. Salomon and G. Turinici in [116]. Liquidity issues were investigated in [148]. Energy markets and power grid management were considered in [15, 127]. Systemic risk was examined in [143]. Price formation was considered in [120, 30, 29, 35, 34, 25]. In [4], the authors develop MFG methods for a wide range of economic problems modeled by partial differential equations. In particular, the material on that paper was the inspiration for part of our discussion in Chapter 3. Further results are reported in [7], [135] and [145].

1.3 Outline of the book

In this book, we have attempted to illustrate the main techniques and methods in mean-field games theory, in several simplified models motivated by economic considerations. We do not claim that any of them reflects the reality accurately. In fact, we secretly hope and encourage our readers to criticize our models and to attempt to improve and develop them. However, we expect the methods and principles put forth here to be useful in the description of heterogeneous agent problems.

The structure of this book reflects these guiding principles. We have divided the book into two parts. In the first one, we develop deterministic models. Little background is necessary beyond calculus and differential equations, although a certain level of mathematical maturity may be required. In the second part, we present supplementary material on stochastic models. Here, a reasonable familiarity with Itô calculus is a prerequisite.

We begin the first part, in Chapter 2, with a brief discussion of deterministic mean-field games. Then, in Chapter 3, we consider simplified problems with heterogeneous agents. In Chapter 4, we propose various models to study economic growth. We address the price formation in an economy with consumer and capital goods, the role of central banks in determining interest rates, international trade and trade imbalances and price impact. We end this first part of the book, in Chapter 5, with an outline of mathematical methods and some theoretical results.

The second part of the book comprises three main chapters, together with a last chapter discussing mean-field games with correlations. In Chapter 6, we present preliminary material on second-order mean-field games. These include elementary notions in stochastic optimal control theory and Hamilton-Jacobi equations, along with topics on the Fokker-Planck equation. Chapter 7 examines growth models in the presence of stochastic shocks. The mathematical analysis of a growth model with noise is reported in Chapter 8. We prove a Verification Theorem and establish monotonicity and concavity properties for the value function. A comparison principle in regulated markets closes the chapter. In Chapter 9 we discuss mean-field games with correlated noise terms. Following some introductory material, we consider a growth model and derive the associated master equation.

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Further bibliographical notes For the reader interested in the history of economic ideas, we suggest the books by Blaug [28] and Heilbroner [100]. An account of the Marginal Revolution can be found in [110]. For a discussion of the contribution of L. Walras and A. Cournot, we refer the reader to [28, Chapter 13], [67] and [146]. Readers interested in a comprehensive account of the contribution of W. S. Jevons and C. Menger, are referred to [28, Chapter 8] and [110, Chapters VI-VIII].

For the first work of J. von Neumann on game theory, we refer the reader to [153]. The book by the author, in collaboration with

O. Morgenstern, [154], is a classical reference in the field. For the breakthrough contribution of J. Nash to the theory of bargaining, see [130].

For mean-field games theory, we suggest the video lectures by P.-L. Lions at the Collège de France [124]. For the first ideas and results in MFG, the references are the seminal papers by J.-M. Lasry and P.-L. Lions [118, 119, 120] and M. Huang, P. Caines and R. Malhamé [103, 106]. The survey [42], [87] and the monograph [21] synthesize recent advances in the field.

Part I

Deterministic models

2

First-order mean-field games

Here, we put forward an outline of the theory of deterministic mean-field games. We begin with a brief review of the deterministic optimal control theory. Next, we proceed with a discussion of the essential properties of the transport equation. Then, we present a typical formulation of first-order mean-field game problems.

2.1 Deterministic optimal control problems

The standard setting of deterministic optimal control problems is the following: consider a single-agent whose state is denoted by $\mathbf{x}_t \in \mathbb{R}^d$, for $t_0 \leq t \leq T$, where $T > 0$ is arbitrarily fixed; the case $T < \infty$ is referred to as the finite horizon optimal control problem.

The state $\mathbf{x}_t$ is governed by the ordinary differential equation

$$\dot{\mathbf{x}}_t = f(\mathbf{v}_t, \mathbf{x}_t), \tag{2.1}$$

where $f : \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}^d$ is a given function and $\mathbf{v}_t \in \mathbb{R}^m$ is a control. At each instant, the agent controls her state $\mathbf{x}_t$ through f, by choosing different values of $\mathbf{v}_t$. To be rigorous, we set $\mathcal{W} \doteq$

$\{\mathbf{v} : [t_0, T] \to W,\ \mathbf{v} \in X\}$, where $W \subseteq \mathbb{R}^m$ and X is a suitable function space (e.g., $L^\infty([t_0, T], W)$).

The agent has certain preferences, expressed through a utility functional $J(\mathbf{v}_t, \mathbf{x}_t; t)$; for $t \in [t_0, T)$, this functional is given by:

$$J(\mathbf{v}, \mathbf{x}; t) \;=\; \int_t^T u(\mathbf{v}_s, \mathbf{x}_s) ds \;+\; \Psi(\mathbf{x}_T),$$

where $u : \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}$ is called the instantaneous utility function of the agent and $\Psi : \mathbb{R}^d \to \mathbb{R}$ is the terminal payoff. The instantaneous utility is also referred to as the running cost. If $T = \infty$, we introduce a discount rate $\beta > 0$ and consider

$$J_\beta(\mathbf{v}, \mathbf{x}; t) = \int_t^{+\infty} e^{-\beta(s-t)} u(\mathbf{v}_s, \mathbf{x}_s) ds. \tag{2.2}$$

Throughout this chapter, we focus on a variant of the problem (2.2), where

$$J_\alpha(\mathbf{v}, \mathbf{x}; t) = \int_t^T e^{-\alpha(s-t)} u(\mathbf{v}_s, \mathbf{x}_s) ds + e^{-\alpha(T-t)} \Psi(\mathbf{x}_T). \tag{2.3}$$

The single-agent seeks to maximize J_α among all possible controls; the supremum over all controls:

$$V(x, t) = \sup_{\mathbf{v} \in \mathcal{W}} J_\alpha(\mathbf{v}, x; t), \tag{2.4}$$

is called the *value function* associated with the deterministic optimal control problem (2.1)-(2.3).

For technical reasons, we make certain assumptions on f, to ensure that the problem can be addressed rigorously. In the remainder of this chapter, we suppose $f \in \mathcal{C}(\mathbb{R}^m \times \mathbb{R}^d, \mathbb{R}^d)$ and

$$|f(v, x_1) - f(v, x_2)| \;\le\; C_\theta \, |x_1 - x_2| \,,$$

for every x_1, $x_2 \in \mathbb{R}^d$ and $v \in \mathbb{R}^m$, with $|v| \le \theta$. Finally, we require f to be linear with respect to v.

If $V(x, t)$ is of class $\mathcal{C}^1(\mathbb{R}^d \times [t_0, T])$, it solves a nonlinear partial differential equation known as the Hamilton-Jacobi equation (HJ). It is given by

$$V_t(x, t) \;-\; \alpha V(x, t) \;+\; H(x, D_x V(x, t)) \;=\; 0, \tag{2.5}$$

where the Hamiltonian $H = H(x, p)$ is defined by

$$H(x,p) \doteq \sup_{v \in W} \left[p \cdot f(v,x) + u(v,x) \right]. \tag{2.6}$$

In the usual language of the optimal control theory, (2.6) is called the Legendre transform of the running cost u. Moreover, as we will prove later, the optimal control $\mathbf{v}_t^*$ is determined by

$$H_p(\mathbf{x}^*, D_x V(\mathbf{x}^*, t)) = f(\mathbf{v}^*, \mathbf{x}^*). \tag{2.7}$$

By solving (2.7) with respect to $\mathbf{v}^*$, we obtain an optimal control in feedback form.

The previous discussion can be made rigorous by means of a Verification Theorem that we present in this chapter.

Dynamic programming principle We begin by discussing a preliminary result, namely, the Dynamic Programming Principle (DPP).

Proposition 2.1.1 (Dynamic Programming Principle)**.** *Let $r \in (t_0, T)$ be fixed and assume that V is defined as in* (2.4)*. Then, we have*

$$V(x,t) = \sup_{\mathbf{v} \in \mathcal{W}} \left(\int_t^r e^{-\alpha(s-t)} u(\mathbf{v}_s, \mathbf{x}_s) ds + e^{-\alpha(r-t)} V(\mathbf{x}_r, r) \right).$$

Proof. The proof has two steps.

Step 1 - For $t \le s \le r$, choose a trajectory $\tilde{\mathbf{x}}$ and let $\tilde{\mathbf{v}}$ be the associated control. Set $y \doteq \tilde{\mathbf{x}}_r$. Then,

$$V(y,r) = \sup_{\mathbf{v} \in \mathcal{W}} \left(\int_r^T e^{-\alpha(s-r)} u(\mathbf{v}_s, \mathbf{x}_s) ds + e^{-\alpha(T-r)} \Psi(\mathbf{x}_T) \right). \tag{2.8}$$

Fix $\epsilon > 0$ and choose an ϵ-optimal control $\mathbf{v}^\epsilon$ for (2.8); as usual, denote by $\mathbf{x}^\epsilon$ the associated trajectory. Then,

$$V(y,r) - \epsilon \le \int_r^T e^{-\alpha(s-r)} u(\mathbf{v}_s^\epsilon, \mathbf{x}_s^\epsilon) ds + e^{-\alpha(T-r)} \Psi(\mathbf{x}_T^\epsilon).$$

Define the control

$$\bar{\mathbf{v}}_s \doteq \begin{cases} \tilde{\mathbf{v}}_s & \text{if } s \in [t, r], \\ \mathbf{v}_s^\epsilon & \text{if } s \in (r, T]. \end{cases}$$

Then,

$$\begin{aligned} V(x,t) &\geq \int_t^T e^{-\alpha(s-t)} u(\bar{\mathbf{v}}_s, \bar{\mathbf{x}}_s) ds + e^{-\alpha(T-t)} \Psi(\bar{\mathbf{x}}_T) \\ &= \int_t^r e^{-\alpha(s-t)} u(\tilde{\mathbf{v}}_s, \tilde{\mathbf{x}}_s) ds + \int_r^T e^{-\alpha(s-t)} u(\mathbf{v}_s^\epsilon, \mathbf{x}_s^\epsilon) ds \\ &\quad + e^{-\alpha(T-t)} \Psi(\mathbf{x}_T^\epsilon) \\ &\geq \int_t^r e^{-\alpha(s-t)} u(\tilde{\mathbf{v}}_s, \tilde{\mathbf{x}}_s) ds + e^{-\alpha(r-t)} \left(V(y,r) - \epsilon \right). \end{aligned}$$

Because $\epsilon > 0$ is arbitrary, we have

$$V(x,t) \geq \int_t^r e^{-\alpha(s-t)} u(\tilde{\mathbf{v}}_s, \tilde{\mathbf{x}}_s) ds + e^{-\alpha(r-t)} V(y,r). \tag{2.9}$$

Taking the supremum with respect to $\tilde{\mathbf{v}}$ in the right-hand side of (2.9), one obtains

$$V(x,t) \geq \sup_{\mathbf{v} \in \mathcal{W}} \left(\int_t^r e^{-\alpha(s-t)} u(\mathbf{v}_s, \mathbf{x}_s) ds + e^{-\alpha(r-t)} V(y,r) \right). \tag{2.10}$$

To conclude the proof, we verify the reverse inequality.

Step 2 - We consider the same trajectory $\tilde{\mathbf{x}}$ and fix $\rho > 0$. By choosing a ρ-optimal control $\mathbf{v}^\rho$, we notice that

$$\begin{aligned} V(x,t) - \rho &\leq \int_t^T e^{-\alpha(s-t)} u(\mathbf{v}_s^\rho, \mathbf{x}_s^\rho) ds + e^{-\alpha(T-t)} \Psi(\mathbf{x}_T^\rho) \\ &= \int_t^r e^{-\alpha(s-t)} u(\mathbf{v}_s^\rho, \mathbf{x}_s^\rho) ds + \int_r^T e^{-\alpha(s-t)} u(\mathbf{v}_s^\rho, \mathbf{x}_s^\rho) ds \\ &\quad + e^{-\alpha(T-t)} \Psi(\mathbf{x}_T^\rho) \\ &\leq \int_t^r e^{-\alpha(s-t)} u(\mathbf{v}_s^\rho, \mathbf{x}_s^\rho) ds + e^{-\alpha(r-t)} V(y,r), \end{aligned}$$

where $\mathbf{x}_s^\rho$ is the trajectory associated with $\mathbf{v}_s^\rho$. Again, because ρ is arbitrary, by taking the maximum, we obtain

$$V(x,t) \leq \sup_{\mathbf{v}\in\mathcal{W}} \left(\int_t^r e^{-\alpha(s-t)} u(\mathbf{v}_s, \mathbf{x}_s) ds + e^{-\alpha(r-t)} V(y,r) \right). \tag{2.11}$$

By gathering (2.10) and (2.11), one concludes the proof. □

A Verification Theorem The DPP is instrumental in the analysis of optimal control problems. By recurring to the DPP, one can rigorously verify that the value function V associated with (2.1)-(2.3) satisfies the Hamilton-Jacobi equation in (2.5), provided V has enough regularity.

In general, the value function is not differentiable. Nevertheless, using the theory of viscosity solutions, it is possible to obtain weaker versions of Theorem 2.1.1, see [66].

Next, we show the converse of that statement, namely: if a function V has enough regularity and it solves the Hamilton-Jacobi equation, then V is the value function of the optimal control problem (2.4).

In our setting, the first step towards a Verification Theorem is to equip (2.5) with a *terminal condition*:

$$V(x,T) = \Psi(x), \tag{2.12}$$

where Ψ accounts for the terminal cost in (2.3).

Theorem 2.1.1 (Verification Theorem)**.** *Let $U \in \mathcal{C}^1(\mathbb{R}^d \times [t_0,T])$ be a solution of* (2.5) *equipped with the terminal condition* (2.12)*. Then,*

$$U(x,t) \geq V(x,t), \quad \text{for every } (x,t) \in \mathbb{R}^d \times [t_0,T].$$

Moreover, if there exists $\mathbf{v}^*$ such that

$$H(\mathbf{x}_t^*, DU(\mathbf{x}_t^*,t)) = DU(\mathbf{x}_t^*,t) \cdot f(\mathbf{v}_t^*, \mathbf{x}_t^*) + u(\mathbf{v}_t^*, \mathbf{x}_t^*), \tag{2.13}$$

where $\mathbf{x}_s^*$ is the trajectory associated with the control $\mathbf{v}^*$, then $\mathbf{v}^*$ is an optimal control for (2.1)-(2.3) *and $U(x,t) = V(x,t)$.*

Proof. We start by noticing that

$$e^{-\alpha(T-t)}U(\mathbf{x}_T,T) = \int_t^T e^{-\alpha(s-t)}\left(-\alpha U + U_x\cdot f(\mathbf{v}_s,\mathbf{x}_s) + U_s\right)ds \\ + U(\mathbf{x}_t,t). \tag{2.14}$$

By combining (2.14) with (2.6), we get

$$U(x,t) \geq \int_t^T e^{-\alpha(s-t)}u(\mathbf{v}_s,\mathbf{x}_s)ds + e^{-\alpha(T-t)}\Psi(\mathbf{x}_T). \tag{2.15}$$

This gives the first part of the Theorem.

For the second part, we notice that (2.13) and (2.14), along with the assumption regarding the terminal condition, yield equality in (2.15). This concludes the proof. □

The Pontryagin maximum principle We now present the Pontryagin maximum principle. This result formalizes a set of necessary conditions for the occurrence of a maximum (or a minimum) in the context of an optimal control problem. Here, we consider the problem given by (2.1) and (2.3).

We first introduce the *adjoint variable*. Let $\mathbf{v}^*$ be an optimal control for (2.1)-(2.3) and denote by $\mathbf{x}^*$ the associated trajectory. The adjoint variable, $\mathbf{q}:\mathbb{R}\to\mathbb{R}^d$, is the solution of

$$\frac{d}{ds}\left(e^{-\alpha(s-t)}\dot{\mathbf{q}}_j(s)\right) = -e^{-\alpha(s-t)}\sum_{i=1}^d \frac{\partial}{\partial x_j} f_i(\mathbf{v}_s^*,\mathbf{x}_s^*)\mathbf{q}_i(s) \\ - e^{-\alpha(s-t)}\frac{\partial}{\partial x_j}u(\mathbf{v}_s^*,\mathbf{x}_s^*), \tag{2.16}$$

with

$$\mathbf{q}(T) = D\Psi(\mathbf{x}_T^*). \tag{2.17}$$

To proceed, we also need the notion of strongly approximately continuous function; we say that a function $\mathbf{v}_s : [t,T]\to\mathbb{R}^m$ is strongly approximately continuous at $s\in[t,T]$ if

$$g(\mathbf{v}_s) = \lim_{\delta\to 0}\frac{1}{\delta}\int_s^{s+\delta} g(\mathbf{v}_r)dr,$$

for all $g \in \mathcal{C}(\mathbb{R}^m)$. In what follows we assume the controls, and their respective trajectories, to be strongly approximately continuous a.e. in $[t, T]$.

Theorem 2.1.2 (Pontryagin maximum principle)**.** *Let* $\mathbf{v}^*$ *be an optimal control for the optimal control problem* (2.1)-(2.3)*. Assume that* $\mathbf{q}$ *is a solution to* (2.16) *equipped with the terminal condition* (2.17)*. Then,*

$$\mathbf{q}(s) \cdot f(\mathbf{v}^*_s, \mathbf{x}^*_s) + u(\mathbf{v}^*_s, \mathbf{x}^*_s) = H(\mathbf{x}^*, \mathbf{q}(s)).$$

Proof. Fix $s \in [t, T)$ such that $\mathbf{v}^*$ is strongly approximately continuous in s, and choose $\delta > 0$ such that $s + \delta \in [t, T]$. Define

$$\mathbf{v}_\delta(r) = \begin{cases} \nu & r \in [s, s+\delta], \\ \mathbf{v}^*_r & \text{otherwise.} \end{cases}$$

We denote by $\mathbf{x}_\delta$ the trajectory associated with $\mathbf{v}_\delta$. Because $\mathbf{v}^*$ is optimal, we have $J_\alpha(\mathbf{v}^*, \mathbf{x}^*; t) \geq J_\alpha(\mathbf{v}_\delta, \mathbf{x}_\delta; t)$. Hence,

$$\begin{aligned} 0 \leq \frac{1}{\delta} \int_s^{s+\delta} & e^{-\alpha(r-t)} \left(u(\mathbf{v}^*, \mathbf{x}^*) - u(\nu, \mathbf{x}_\delta) \right) dr \\ + \frac{1}{\delta} \int_{s+\delta}^T & e^{-\alpha(r-t)} \left(u(\mathbf{v}^*, \mathbf{x}^*) - u(\mathbf{v}^*, \mathbf{x}_\delta) \right) dr \\ + \frac{1}{\delta} & e^{-\alpha(T-t)} \left(\Psi(\mathbf{x}^*(T)) - \Psi(\mathbf{x}_\delta(T)) \right). \end{aligned} \tag{2.18}$$

We proceed by taking the limit $\delta \to 0$ in (2.18). Because $\mathbf{v}^*$ is strongly approximately continuous, the first term on the right-hand side of (2.18) becomes

$$e^{-\alpha(s-t)} \left(u(\mathbf{v}^*, \mathbf{x}^*) - u(\nu, \mathbf{x}^*) \right),$$

as $\delta \to 0$. By recurring to the mean-value theorem, the second term becomes

$$\int_{s+\delta}^T e^{-\alpha(r-t)} \int_0^1 u_x(\mathbf{v}^*, \mathbf{x}^* + \delta\lambda\xi_\delta(r)) \cdot \xi_\delta(r) d\lambda dr,$$

where

$$\xi_\delta(r) \doteq \frac{\mathbf{x}^* - \mathbf{x}_\delta}{\delta}.$$

Notice that $\xi_\delta \to \xi$ as $\delta \to 0$, where ξ solves the following Cauchy problem:

$$\begin{cases} \dot{\xi}(r) = f_x(\mathbf{v}^*, \mathbf{x}^*)\xi(r) \quad r \geq s, \\ \xi(s) = f(\mathbf{v}^*, \mathbf{x}^*) - f(\nu, \mathbf{x}^*). \end{cases} \tag{2.19}$$

Therefore, by taking the limit $\delta \to 0$, (2.18) yields the inequality:

$$0 \leq e^{-\alpha(s-t)} \left(u(\mathbf{v}^*, \mathbf{x}^*) - u(\nu, \mathbf{x}^*)\right) + \int_s^T e^{-\alpha(r-t)} u_x(\mathbf{v}^*, \mathbf{x}^*) \cdot \xi(r) dr \tag{2.20}$$
$$+ e^{-\alpha(T-t)} D\Psi(\mathbf{x}_T^*) \cdot \xi(T).$$

Consider now

$$\frac{d}{dr}\left[e^{-\alpha(r-t)}q(r)\xi(r)\right] = \frac{d}{dr}\left(e^{-\alpha(r-t)}q(t)\right)\xi(t) + e^{\alpha(r-t)}q(t)\dot{\xi}(t);$$

by combining (2.19) with (2.16), we obtain

$$\frac{d}{dr}\left[e^{-\alpha(r-t)}q(r)\xi(r)\right] = -e^{-\alpha(r-t)}u_x(\mathbf{v}^*, \mathbf{x}^*)\xi(r).$$

This, along with (2.20) and the definition of the Hamiltonian H, concludes the proof. □

In the literature, (2.17) is called a *transversality* condition. Other related transversality conditions are typical in optimal control problems. For a more detailed discussion, see [66].

Next, we briefly present the very basics concerning the transport equation and discuss a few facts about its solutions.

2.2 Transport equation

Consider a population (of economic agents, bacteria, molecules, etc.) whose microscopic dynamics, that is, at the single-agent level, is governed by the following ordinary differential equation:

$$\begin{cases} \dot{\mathbf{x}}_t = b(\mathbf{x}_t, t) \quad t > t_0, \\ \mathbf{x}_{t_0} = x, \end{cases} \tag{2.21}$$

where b is a Lipschitz vector field. The vector field b induces a flow that transports the population's density, pushing it forward or backward in time. The evolution of this density, denoted by ρ, is described by the first-order partial differential equation

$$\rho_t + \operatorname{div}(b(x,t)\rho) = 0. \tag{2.22}$$

That is called transport equation. Next, we present a few elementary facts about (2.22) that are critical to its physical interpretation.

Proposition 2.2.1. *Let ρ be a solution of* (2.22) *with initial condition $\rho(x,t_0) = \rho_0(x)$, for some probability density $\rho_0 \in \mathcal{C}_c^\infty(\mathbb{R}^d)$. Then we have:*

(Positivity of solutions) $\rho(x,t) \geq 0$ *for all $t > t_0$;*

(Mass conservation)

$$\int_{\mathbb{R}^d} \rho(x,t)dx = \int_{\mathbb{R}^d} \rho_0(x)dx = 1,$$

for all $t > t_0$.

Proof. The first claim follows from the maximum principle, since $\rho_0 \geq 0$. To verify the second claim, notice that

$$\frac{d}{dt}\int_{\mathbb{R}^d} \rho(x,t)dx = 0$$

for all $t > t_0$. This, and the fact that

$$\int_{\mathbb{R}^d} \rho_0(x)dx = 1,$$

concludes the proof. □

The vector field b in (2.21) generates a flow in $\mathbb{R}^d$ denoted by Φ_t. This flow maps $x \in \mathbb{R}^d$ to the solution of (2.21) at the instant $t > t_0$, with initial condition x.

Let $\theta(x,t)$ be the measure on $\mathbb{R}^d$ given by

$$\int_{\mathbb{R}^d} \phi(x)\theta(x,t)dx = \int_{\mathbb{R}^d} \phi\left(\Phi_t(x)\right)\theta_0 dx. \tag{2.23}$$

Then, we have the following:

Proposition 2.2.2. *Let θ be the measure on $\mathbb{R}^d$ determined by* (2.23)*. Assume that the vector field b is Lipschitz continuous and denote by Φ_t the flow corresponding to* (2.21)*. Then,*

$$\begin{cases}\theta_t(x,t) + \operatorname{div}(b(x,t)\theta(x,t)) = 0, & (x,t) \in \mathbb{R}^d \times [t_0,\infty)\\ \theta(x,t_0) = \theta_0(x), & x \in \mathbb{R}^d,\end{cases} \tag{2.24}$$

in the distributional sense.

Proof. We start by recalling the definition of solution in the distributional sense. A function ρ is said to solve (2.24) in the distributional sense if

$$-\int_{t_0}^{\infty}\int_{\mathbb{R}^d} \rho(x,t)\left(\phi_t(x,t) + b(x,t)\phi_x(x,t)\right) dxdt = \int_{\mathbb{R}^d} \rho_0(x)\phi(x,t_0)dx,$$

for every $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^d \times [t_0,\infty))$. Differentiate both sides of (2.23) with respect to t to obtain

$$\int_{\mathbb{R}^d} \phi(x)\theta_t(x,t)dx = \int_{\mathbb{R}^d} \left(b(\Phi_t(x),t)\phi_x\left(\Phi_t(x)\right)\right)\theta_0(x)dx.$$

Therefore,

$$\int_{\mathbb{R}^d} \phi(x)\theta_t(x,t)dx = \int_{\mathbb{R}^d} \left(b(x,t)\phi_x(x)\right)\theta(x,t)dx,$$

where, we have used the definition of Φ_t. Integration by parts concludes the proof. □

We conclude this section with a result relating (2.21) with the Dirac delta evaluated along $\mathbf{x}(t)$.

Proposition 2.2.3. *Assume that* (2.21) *holds and suppose that b is Lipschitz continuous. Then, $\delta_{\Phi_t(x)}$ solves* (2.24)*, in the distributional sense.*

Proof. If $\rho_0(x) \equiv \delta_{x_0}$, the right-hand side of (2.2) becomes

$$\int_{\mathbb{R}^d} \delta_{x_0}(x)\phi(x,t_0)dx = \phi(x_0,t_0).$$

Moreover, if $\rho(x,t) = \delta_{\Phi_t}(x)$, we have:

$$-\int_{t_0}^{\infty}\int_{\mathbb{R}^d} \delta_{\Phi_t(x_0)}(x)\left(\phi_t(x,t) + b(x,t)\phi_x(x,t)\right)dxdt$$
$$= -\int_{t_0}^{\infty} \phi_t(\Phi_t(x_0),t) + b(\Phi_t(x_0),t)\phi_x(\Phi_t(x_0),t)dt.$$

Because of (2.21), we then have

$$\begin{aligned}-\int_{t_0}^{\infty}\int_{\mathbb{R}^d} \delta_{\Phi_t(x_0)}(x)\left(\phi_t(x,t) + b(x,t)\phi_x(x,t)\right)dxdt \\ = -\int_{t_0}^{\infty} \frac{d}{dt}\left[\phi\left(\Phi_t(x_0),t\right)\right] \\ = \phi(x_0,t_0),\end{aligned}$$

which closes the proof. □

Remark 2.2.1. *The converse of Proposition 2.2.3 is also valid. To verify it, it suffices to observe that, if $\delta_{\Phi_t(x)}$ is a solution of (2.24) in the distributional sense, the previous proof implies*

$$\int_{t_0}^{\infty} \left(b(\Phi_t(x_0),t) - \dot{\Phi}_t(x_0)\right)\phi_x(\Phi_t(x_0),t)dt = 0,$$

for every $\phi \in \mathcal{C}_c^{\infty}(\mathbb{R}^d \times [t_0,\infty))$.

2.3 Reduced mean-field games

As mentioned in the Introduction, among the main motivations for the study of mean-field games are the connections with the theory of N-players differential games. A solution of the MFG formalizes for the case $N \to \infty$ the concept of Nash equilibrium of the N-person differential games.

Some reduced mean-field game models can be rigorously derived as the limit of the equations characterizing N-players differential games, when the limit $N \to \infty$ is considered. In the sequel, we present a heuristic derivation of a model mean-field game.

We start by describing the single-agent optimal control problem. Suppose that the state of a certain agent at time t is characterized by the vector $\mathbf{x}_t \in \mathbb{R}^d$. In addition, assume that the state $\mathbf{x}_t$ is governed by the following ordinary differential equation:

$$\begin{cases} \dot{\mathbf{x}}_t = \mathbf{v}_t, & t \in (t_0, T] \\ \mathbf{x}_{t_0} = x, \end{cases} \tag{2.25}$$

where $\mathbf{v}_t$ is a control and $x \in \mathbb{R}^d$ is a given initial condition. This agent seeks to maximize the functional:

$$J(\mathbf{v}, x; t_0) \doteq \int_{t_0}^{T} u(\mathbf{v}_s, \mathbf{x}_s) + g[\rho] ds + u_T(\mathbf{x}_T), \tag{2.26}$$

where u is a Lagrangian, u_T is a terminal condition and $g[\rho]$ is a term to be made precise later. Define

$$H(x,p) \doteq \sup_v \left(p \cdot v + u(v,x) \right).$$

We know, from Section 2.1, that, under regularity assumptions, the value function V associated with the optimal control (2.25)-(2.26) solves the Hamilton-Jacobi equation:

$$\begin{cases} V_t(x,t) + H(x, D_x V(x,t)) = g[\rho], & \text{in } \mathbb{R}^d \times [t_0, T) \\ V(x,T) = u_T(x), & \text{in } \mathbb{R}^d. \end{cases}$$

Furthermore, the optimal control $\mathbf{v}^*$ is given in feedback form by

$$\mathbf{v}^* = H_p(\mathbf{x}, D_x V(\mathbf{x}, t)).$$

If this agent is rational, her state evolves according to (2.25), where the velocity $\mathbf{v}$ is given by the feedback optimal control $\mathbf{v}^*$.

Next, we consider a population of rational and indistinguishable agents. The statistical distribution of agents in this population is encoded in the density ρ of a probability measure. In this population, each agent faces the same optimal control problem. Therefore, the state of each individual is driven by

$$\begin{cases} \dot{\mathbf{x}}_s = H_p(\mathbf{x}, D_x V(\mathbf{x}, s)), \\ \mathbf{x}_{t_0} = x. \end{cases} \tag{2.25'}$$

By the previous discussion, g in (2.26) encodes the dependence of the agents' cost functional on the density of the entire population. Accordingly, the optimization problem faced by the agents depends on the evolution of ρ. Conversely, ρ evolves according to a vector field determined by V.

From Section 2.2, we know that the density ρ evolves according to:

$$\begin{cases} \rho_t(x,t) + \operatorname{div}(H_p(x, D_xV(x,t))\rho(x,t)) = 0, & \text{in } \mathbb{R}^d \times (t_0, T] \\ \rho(x,t_0) = \rho_0(x), & \text{in } \mathbb{R}^d. \end{cases}$$

The mean-field game system associated with this problem is:

$$\begin{cases} V_t(x,t) + H(x, D_xV) = g[\rho](x,t), & \text{in } \mathbb{R}^d \times [t_0, T) \\ \rho_t(x,t) + \operatorname{div}(H_p(x, D_xV)\rho(x,t)) = 0, & \text{in } \mathbb{R}^d \times (t_0, T], \end{cases} \tag{2.27}$$

equipped with initial-terminal boundary conditions

$$\begin{cases} V(x,T) = u_T(x), & \text{in } \mathbb{R}^d. \\ \rho(x,t_0) = \rho_0(x), & \text{in } \mathbb{R}^d. \end{cases} \tag{2.28}$$

Roughly speaking, the model problem (2.27)-(2.28) accounts for the mutual dependence of two distinct processes. First, the evolution of a population that behaves optimally; second, an optimization problem population-dependent. It is precisely this interplay that characterizes the optimal control $\mathbf{v}^* = H_p(\mathbf{x}, D_xV(\mathbf{x},t))$ as a Nash equilibrium.

In this model problem, the dependence of the value function on the density is encoded by the map g. However, distinct settings may arise; for example, this dependence can be encoded in the utility function u, as in the formulation of the congestion problem, see [72], or through equilibrium conditions. This last type of coupling is of paramount importance in economic problems and will be detailed in next chapters.

Bibliographical notes For the material presented in Section 2.1, we refer the reader to [17], [66] and [126]. The definition of strongly approximately continuous function can be found in [60]. A comprehensive account of the theory of viscosity solutions of first-order

Hamilton-Jacobi equations can be found in [17] and [66]. For a more detailed discussion of the Pontryagin maximum principle, see [66].

For an introduction to the transport equation, see [58]. Regarding applications to mathematical biology, see [138]. In [138, Chapter 6], one can find a self-contained account of mathematical methods for the transport equation. These range from elementary properties to the Di Perna-Lions theory and more sophisticated techniques.

Further references on reduced mean-field games can be found in [42], [124] and [87].

3

Some simple economic models

In this chapter, we put forward three models with heterogeneous agents. These describe simplified problems, that, we believe, depict the key techniques we develop here. We begin with a version of the Solow model for capital accumulation. We use this model to illustrate some differences between free market economies and centrally planned ones. Strictly speaking, the free-market version of the Solow model is not a mean-field game as there are no interactions between the agents. The planned economy version turns out to be a mean-field control problem. Next, we consider a deterministic Aiyagari-Bewley-Huggett model that includes both wages and capital accumulation.

3.1 Economic models

3.1.1 Agents and their states

Agents in the economy are divided into two main groups: microeconomic agents and macroeconomic agents.

Microeconomic agents Microeconomic agents are small participants in the economy and, individually, cannot influence the economic outcomes. Consumers, workers, and firms are examples of microeconomic agents.

Macroeconomic agents In contrast to their microeconomic counterparts, macroeconomic agents are able to impact directly the economy. Instances of macroeconomic agents include central banks, large firms (e.g., monopolistic firms) or governments.

Microeconomic variables Microeconomic agents in the economy are characterized by individual state variables that we call microeconomic variables. These include, for instance, the wealth, capital or wages of each small participant in the economy. A probability measure ρ describes the global distribution of agents along the state space of the economy.

Heterogeneity in the microeconomic variables is the hallmark of heterogeneous agent models. The density ρ synthesizes all the information concerning the agents' distribution.

Macroeconomic variables Unlike microeconomic variables, that are heterogeneous among the economy, a macroeconomic variable is a global quantity of the system. These include exogenous variables such as taxes, external demand for goods, or variables that are determined by equilibrium conditions such as relative prices.

3.1.2 Dynamics and controls

We start by describing the controls of the agents in the economy.

Controls (actions) Agents in the model are allowed to choose the levels of certain quantities. These quantities are called *controls* and, in the framework of mean-field games, will be also referred to as *strategies*.

Controls can be either of micro or macroeconomic nature. The former case regards actions taken by microeconomic agents; these concern consumption levels, investment, etc. On the other hand, macroeconomic controls concern the actions of the macroeconomic agents in the economy.

Constitutive relations The relations between various quantities in the economy are specified by constitutive relations that are model-dependent. An example is the relation between productivity and capital or the capital depreciation function.

Dynamics Controls and the constitutive relations, determine the evolution of the agents' state in the economy. This is formalized by certain prescribed laws, that relate those objects with the state variables. The collection of these laws is known as the *dynamic* of the model.

This dynamic can be deterministic or include stochastic elements such as diffusions or jump processes.

3.1.3 Preferences

The assumption of rationality presumes that agents choose controls in order to optimize some functional. This functional encodes the preferences of the agents with respect to the variables of the model.

For example, it is reasonable to suppose that a microeconomic agent in the model prefers more consumption to less, whereas a central bank aims at controlling price levels while promoting growth and capital accumulation. In the case of microeconomic agents, these ideas are formalized through an instantaneous utility function. Monotonicity and concavity encode many qualitative properties of the model. For instance, with respect to consumption, the utility function is increasing (agents are assumed to prefer more consumption to less) and concave, for the satisfaction, due to increments in consumption, is decreasing. In case of a macroeconomic agent, preferences are represented by an instantaneous welfare function.

Because agents make choices over a particular time span, we introduce the intertemporal counterparts of the utility and the welfare functions.

3.1.4 Equilibrium conditions

The economic models in this book are constrained by conditions relating the various quantities involved in their formulation. These constraints are called *equilibrium conditions* and formalize the intuitive notions behind the economic processes described in the models.

An example of such a condition is the following: in the setting of a closed economy, the total investment must equal the total production of capital goods.

3.2 Solow model with heterogeneous agents

Here, we describe a heterogeneous agent capital accumulation model. This model serves to illustrate two distinct economies - a free market economy where the price level is determined by a market clearing condition, and a centrally planned economy, in which the central planner sets the share of production that is consumed by the agents. We proceed by detailing the basic elements of this model.

Microeconomic agents Agents in the economy are characterized by their levels of capital, $k_t \geq 0$. The distribution of agents in the state space is given by a probability density $\rho = \rho(k, t)$.

Constitutive relations The model has a resource constraint, entailed by a production function, $f = f(k)$. This function encodes the technology of the economy. The production function gives the amount of output $f(k)$ of an agent with capital k.

We assume that $f'(k) > 0$, for $k > 0$, and

$$\lim_{k \to 0} f'(k) = +\infty \qquad \text{and} \qquad \lim_{k \to \infty} f'(k) = 0. \tag{3.1}$$

The limits in (3.1) formalize the notion of *marginal productivity of capital.*

The production of an agent with capital k_t is divided between an amount c_t of consumption and i_t of investment. This corresponds to the microeconomic resource constraint:

$$f(k_t) = c_t + i_t, \tag{3.2}$$

where $c_t, i_t \geq 0$.

Dynamics: capital accumulation In this economy, the accumulation of capital is governed by the following ordinary differential equation:

$$\dot{k}_t = i_t - \delta k_t, \tag{3.3}$$

where $0 < \delta < 1$ is the capital depreciation rate of the economy, and i_t is the investment.

Next, we solve this model by making two distinct sets of assumptions. First, we consider a free-market economy; in this setting, agents choose their consumption levels in order to maximize some functional $J = J(c, k, t)$. Later, we present a planned economy, in which the consumption level is chosen by a central planner that maximizes a welfare function of the consumption level of the economy.

3.2.1 Free-market economy

In a free-market economy, agents are free to allocate their production, $f(k_t)$, between consumption and investment in (3.2).

Microeconomic actions (controls) We assume that agents control the amount of consumption, c_t. The agents select the control by maximizing certain preferences, subject to the constraints $c_t \geq 0$, $i_t \geq 0$ and (3.2). From this, we derive that $0 \leq c_t \leq f(k_t)$.

Agents' preferences Agents in the economy are assumed to be rational, i.e., they maximize preferences over quantities of the model. We assume these preferences are represented by a utility function $u = u(c, k)$. As in (3.1), the instantaneous utility function u presents decreasing marginal increments. That is, the map $(c, k) \mapsto u(c, k)$ is increasing in (c, k) and concave. Given a discount rate $\alpha > 0$, agents maximize the intertemporal counterpart of the instantaneous utility function u:

$$J(c_t, k_t, t) = \int_t^\infty e^{-\alpha(s-t)} u(c_s, k_s) ds. \tag{3.4}$$

In this case, agents choose their consumption level according to the optimal control problem comprising (3.3) and (3.4). That is, agents are interested in

$$V(k,t) = \sup_{c_t} \int_t^{\infty} e^{-\alpha(s-t)} u(c_s, k_s) ds;$$

If V is regular enough, it solves the following Hamilton-Jacobi equation:

$$V_t - \alpha V + H(k, V_k) = 0,$$

where $H(k, q_k)$ is given by

$$H(k, V_k) = \sup_{0 \leq c \leq f(k)} \left((f(k) - c - \delta k) V_k + u(c,k) \right). \tag{3.5}$$

The optimal control c^*, when it exists, is given in feedback form by

$$H_q(k,q) = f(k) - c^* - \delta k, \tag{3.6}$$

where the second equality follows from (3.2).

Transport of the agents' density The density of agents in the state space is governed by a transport equation. The assumption of rationality implies that every agent acts according to the optimal control (3.6). Therefore, the evolution of the density ρ is driven by

$$\rho_t(k,t) + \left(H_q(k, V_k)\rho(k,t)\right)_k = 0.$$

Consequently, the Solow growth model for a free-market economy can be written as the system:

$$\begin{cases} V_t - \alpha V + H(k, V_k) = 0 \\ \rho_t(k,t) + \left(H_q(k, V_k)\rho(k,t)\right)_k = 0, \end{cases}$$

where H is given as in (3.5).

3.2.2 Planned economy

As before, we suppose that agents have a production function $f(k)$ but they are not allowed to allocate freely between consumption and investment.

Constitutive relations In this economy, a planner controls the fraction λ_t of the production that can be used for investment or consumption. Hence, the investment and consumption of a typical agent are

$$i_t = \lambda_t f(k_t), \qquad c_t = (1-\lambda_t) f(k_t).$$

Macroeconomic controls The central planner controls the rate λ_t with $0 \leq \lambda_t \leq 1$.

Transport of the agents' density The density of agents in the state space is governed by the transport equation:

$$\rho_t(k,t) + ((\lambda_t - \delta) f(k)\rho(k,t))_k = 0. \tag{3.7}$$

Macroeconomic variables For convenience, we introduce here the aggregate production

$$F_t = \int f(k)\rho(k,t). \tag{3.8}$$

Macroeconomic preferences The central planner aims at maximizing a welfare function that measures the consumption. The planner has an instantaneous welfare function U and a positive discount factor α. The intertemporal consumption functional is

$$\int_t^{+\infty} e^{-\alpha(s-t)} U((1-\lambda_s)F_s) ds. \tag{3.9}$$

Together with the transport equation (3.7), this is a mean-field control problem. We regard ρ as a state variable, with controlled dynamics given by (3.7) and we seek to maximize the objective functional (3.9) that depends on ρ through (3.8).

3.3 The Aiyagari-Bewley-Huggett model: deterministic case

In the sequel, we discuss a deterministic version of the Aiyagari-Bewley-Huggett model. This model describes an economy where agents are heterogeneous in wealth and wage. The interest rate is the single macroeconomic variable, and it is determined by a zero net supply condition.

Microeconomic agents At each instant $t \geq t_0$, agents in the economy are characterized by their income and wealth levels, respectively denoted by z_t and a_t.

For each time t, we assume the agents' statistical properties are encoded in a probability density in $\mathbb{R}^2$ denoted by $\rho = \rho(a, z, t)$.

Macroeconomic variables There is a single macroeconomic variable, the interest rate of the economy, denoted by r_t. This is simultaneously the rate at which agents can borrow or lend wealth, that is, there are no financial intermediaries.

Constitutive relations To simplify, we assume that the evolution of the wage of each agent follows a mean-reverting deterministic dynamic $\dot{z} = -(z_t - \bar{z})$, where $\bar{z}$ represents a reference wage level. To keep the presentation as simple as possible, we set $\bar{z} \equiv 1$. This is a rather arbitrary choice and many other alternatives are possible.

Microeconomic actions (controls) Agents in the economy are assumed to choose their consumption levels, c_t, for $t_0 \leq t \leq T$.

Microeconomic dynamic The state variables characterizing the agents in the economy are governed by two ordinary differential equations. These are the following:

$$\begin{cases} \dot{a}_t = z_t - c_t + r_t a_t \\ \dot{z}_t = -(z_t - 1). \end{cases} \tag{3.10}$$

Microeconomic preferences We assume that agents are rational, i.e., they maximize a certain set of preferences over the variables of the model. In addition, we assume these preferences to be modeled by an instantaneous utility function $u = u(c_t, a_t, z_t, r_t)$. Because we are interested in modeling the economy for the future times $s \geq t$, we consider the intertemporal counterpart of u, given by

$$J(c,a,z;t) = \int_t^\infty e^{-\alpha(s-t)} u(c_s, a_s, z_s, r_s) ds. \tag{3.11}$$

The assumption of rationality yields

$$V(a,z,t) = \sup_{c_t} J(c_t, a_t, z_t; t).$$

When it exists, V is called the value function of the (deterministic) optimal control problem (3.10)-(3.11).

If V is regular enough, it is the solution of the following Hamilton-Jacobi equation:

$$V_t - \alpha V + H(a,z,r,V_a,V_z,t) = 0,$$

where the Hamiltonian H is given by

$$H(a,z,r,V_a,V_z) \; = \; \sup_c \left(V_a \dot{a} + V_z \dot{z} + u \right).$$

Moreover, the optimal control c_t^* is given in feedback form by:

$$\begin{cases} H_{q_a}(a,z,r,V_a,V_z) = z_t - c_t^* + r_t a_t, \\ H_{q_z}(a,z,r,V_a,V_z) = -(z_t - 1). \end{cases}$$

Equilibrium conditions We assume that agents in the economy can borrow and lend at the current interest rate, r_t. To close the model, one must assume that each dollar that is borrowed by an agent was saved by another one. This condition is formalized by requiring that

$$\int a\rho(a,z,t) dadz \; = \; 0; \tag{3.12}$$

the equality in (3.12) is referred to as *zero net supply* condition.

It is clear from (3.12) that r_t depends on the density ρ. Therefore, the interest rate encodes the dependence of V on the density of the population.

Transport of the agents' density Given (3.10), the agents' density $\rho(a,z,t)$ evolves according to the transport equation:

$$\rho_t(a,z,t) + (D_{q_a}H\rho)_a + (D_{q_z}H\rho)_z = 0. \tag{3.13}$$

Meanwhile, agents are assumed to behave rationally, i.e., to choose the optimal consumption level c_t^*. Hence, (3.13) becomes

$$\rho_t(a,z,t) + ((z - c^* + ra)\rho)_a - ((z-1)\rho)_z = 0.$$

Then, the deterministic Aiyagari-Bewley-Huggett model can be written as the following mean-field game system:

$$\begin{cases} V_t - \alpha V + H(a,z,r,V_a,V_z) = 0 \\ \rho_t(a,z,t) + ((z - c^* + ra)\rho)_a - ((z-1)\rho)_z = 0, \end{cases}$$

together with initial conditions for the density ρ.

Analysis of the equilibrium condition The equilibrium condition (3.12) implies

$$0 = \frac{d}{dt}\int a\rho(a,z,t)dadz = -\int (z - c^*)\rho(a,z,t)dadz.$$

Moreover,

$$\frac{d}{dt}\int (z-1)^2\rho dadz = -2\int (z-1)^2\rho dadz.$$

Therefore, as one would expect,

$$\int z\rho dadz \to 1.$$

Thus, we obtain

$$\int c^*\rho(a,z,t)dadz \to, \tag{3.14}$$

as $t \to \infty$. The relation in (3.14) formalizes the intuitive notion that the total consumption in the population must equal the total income.

Bibliographical notes For the material in Section 3.2, we refer the reader to the original paper by R. Solow, [149]. A comprehensive introduction to the modern theory of economic growth is the subject of [1]. For a mean-field games approach to the planned economy, we refer the reader to the paper by G. Nuño and B. Moll [135]. The work of R. Lucas and B. Moll [145] on the growth of knowledge also relates to the topic of this chapter.

The reader interested in topics related to the dynamic macroeconomic theory may find the book by M. Wickens useful [156].

For the original contribution of S. Aiyagari, T. Bewley, and M. Huggett, we refer the interested reader to their seminal papers [10], [27], and [107]. Our discussion in Section 3.3 of the Aiyagari-Bewley-Huggett model as a mean-field game is inspired by [7] and [4]. Compared to these references, the model we present here is simplified in various important points since we did not include in our discussion stochastic effects and boundary conditions.

4

Economic growth and MFG - deterministic models

In this chapter, we examine models of economic growth with heterogeneous agents. We begin with the problem of wealth and capital accumulation, with a capital-dependent production function. Then, we investigate the role of a central bank that controls the economy's interest rate. Afterward, we discuss trade imbalances and international trade. Then, we study the issue of price impact in regulated markets. The model in Section 4.1 is a mean-field game where the coupling between the Hamilton-Jacobi equation and the transport equation is determined by an equilibrium condition. By introducing a macroeconomic agent, a central bank, the model in Section 4.2 becomes an optimal control problem of a mean-field game. Next, the analysis of trade imbalances in Section 4.3 becomes an infinite dimensional game between two agents, whose dynamics is given by two mean-field games. Finally, the issue of price impact in regulated markets, through taxation of trade imbalances, is considered in Section 4.4. This is a mean-field game that depends on a price-impact function (e.g. import duties).

4.1 A growth model with heterogeneous agents

Our first growth model is a simple wealth and capital accumulation problem, with capital-dependent production. Here, we present its basic structures and elements according to the outline in Section 1.2.

Microeconomic agents In the present model, a typical individual in the economy has an amount a_t of consumer goods and k_t units of capital, at time t. Consumer goods are used as the numerary of the economy, in the Walrasian sense, which sets its price equal to the unity.

Both the consumer goods a_t and the capital k_t have natural constraints. Borrowing constraints correspond to the inequality $a_t \geq a_0$, for some $a_0 \leq 0$. Moreover, it is natural to assume that capital is non-negative $k_t \geq 0$. However, to simplify the discussion, we will implement these constraints as soft constraints through a utility function.

The microeconomic distribution function $\rho(a, k, t)$ determines the distribution of agents in the state space. We suppose that $\rho \geq 0$ is normalized, so that

$$\int_{\mathbb{R}^2} \rho(a, k, t) da dk = 1,$$

for all $t \geq 0$.

Macroeconomic variables In our model, the only macroeconomic variable is the (relative) price of capital, denoted by p_t. For convenience, we refer to

$$W_t = \int (a_t + p_t k_t) \, d\rho(a, k, t)$$

as the total *wealth* of the economy.

Constitutive relations We suppose the economy has the following constitutive relations. The first one is a production function $F(k, p)$

that gives the total value (measured in consumer goods) of consumer goods $\Theta(k,p)$ and capital goods $\Xi(k,p)$, produced by an agent with capital k at a price level p. In particular, we have

$$F(k,p) = \Theta(k,p) + p\Xi(k,p).$$

The functions Θ and Ξ take into account that agents are able to change, at least partially, their production from consumer to capital goods and vice-versa. The precise way in which this choice is made is not relevant in this model.

Finally, the depreciation of the capital stock is governed by a function g, depending on both the capital level k and the price p, i.e., $g = g(k,p)$.

Microeconomic actions (controls) We assume further that, at every instant t, each agent controls her consumption and investment levels, c_t and i_t, respectively.

Microeconomic dynamics In the present model, the evolution of the state variables is governed by two ordinary differential equations. The stock of consumer goods is driven by the following:

$$\dot{a}_t = -c_t - p_t i_t + F(k_t, p_t). \tag{4.1}$$

In addition, the stock of capital varies according to

$$\dot{k}_t = g(k_t, p_t) + i_t. \tag{4.2}$$

Microeconomic preferences Agents in the economy have preferences about their consumption and investment levels, stocks of consumer goods and capital, and the price level of the economy. These preferences are represented by an instantaneous utility function $u = u(c_t, i_t, a_t, k_t, p_t)$. Agents seek to maximize the intertemporal counterpart of u, i.e.,

$$V(a,k,t) = \sup_{c_t, i_t} \int_t^\infty e^{-\alpha(s-t)} u(c_s, i_s, a_s, k_s, p_s) ds. \tag{4.3}$$

The assumption of rationality is critical in obtaining (4.3).

Optimal control problem Each agent in the economy faces an optimal control problem. The function $V(a,k,t)$, defined in (4.3), is called the *value function* of the problem (4.1)-(4.3). If V has enough regularity, i.e., it is of class $\mathcal{C}^1$, it solves the following Hamilton-Jacobi equation:

$$V_t(a,k,t) \;-\; \alpha V(a,k,t) \;+\; H(a,k,p,V_a,V_k) = 0, \tag{4.4}$$

where the Hamiltonian H is given by

$$H(a,k,p,V_a,V_k) = \sup_{c,i} \left(V_a \dot{a} + V_k \dot{k} + u \right). \tag{4.5}$$

Moreover, the optimal controls c_t^* and i_t^* are determined in feedback form by the equations

$$\begin{cases} H_{q_a}(a_t,k_t,p_t,V_a,V_k) \;=\; -\,c_t^* \;-\; p_t i_t^* \;+\; F(k_t,p_t), \\ H_{q_k}(a_t,k_t,p_t,V_a,V_k) = \; g(k_t,p_t) \;+\; i_t^*, \end{cases} \tag{4.6}$$

where $H = H(a,k,p,q_a,q_k)$ is given by (4.5).

Transport of the agents' population Now, we discuss the evolution of the agents' density. To this end, we introduce the associated transport equation. Because the agents' state variables are governed by (4.1) and (4.2), the equation that describes the evolution in time of the population's density is a transport equation. As before, we denote by $\rho = \rho(a,k,t)$ the density of the agents.

The assumption that agents act to optimize their preferences is central to the optimal control problem discussed in Section 4.1. However, we note that the evolution of the agents' population depends upon the consumption and investment levels, according to (4.7). Because rationality implies that individuals shall consume c^* and invest i^*, the population evolution is driven by the vector field $(\dot{a},\dot{k})$, evaluated at these values. Then, we have the transport equation for ρ:

$$\rho_t \;+\; \left((-\,c^* \;-p_t i^* \;+F(k))\,\rho\right)_a + \left((g(k,p_t) \;+\; i^*)\,\rho\right)_k = 0. \tag{4.7}$$

Hence, using (4.6), (4.7) becomes

$$\rho_t \;+\; \left(H_{q_a}\rho\right)_a + \left(H_{q_k}\rho\right)_k = 0. \tag{4.8}$$

Equilibrium conditions The equations (4.4) and (4.8) are coupled by a market-clearing condition. This condition requires the aggregate production of capital goods to match the aggregate investment, and is given by

$$\int i d\rho(a,k,t) = \int \Xi(k,p) d\rho(a,k,t), \tag{4.9}$$

for every $t > 0$. This equilibrium condition determines the price level p_t.

A mean field game model The mean-field game formulation of the growth model is given by the coupling of (4.4) and (4.8):

$$\begin{cases} V_t(a,k,t) - \alpha V(a,k,t) + H(a,k,p,V_a,V_k) = 0, \\ \rho_t + (H_{q_a}\rho)_a + (H_{q_k}\rho)_k = 0, \end{cases} \tag{4.10}$$

together with the equilibrium condition (4.9). In (4.10) we prescribe initial conditions for ρ.

4.2 A growth model with a macroeconomic agent

Building upon the previous section, we introduce a growth model with a central bank acting as a macroeconomic agent. The central bank controls the interest rate of the economy. Its aim is to balance between price stability and economic growth.

Microeconomic agents As before, microeconomic agents are characterized by their levels of consumer goods a_t and stock of capital k_t.

Macroeconomic agent The single macroeconomic agent in this economy is a central bank. Its state is determined by a quantity A_t, that represents its assets, and the distribution of microeconomic agents $\rho(a,k,t)$.

Macroeconomic variables As before, the price level of the economy p_t is a macroeconomic variable. In addition, the economy has an interest rate, r_t. This rate is set by the central bank, and, therefore, it is a control variable for this macroeconomic agent.

Constitutive relations The constitutive relations of the economy, namely, the production and depreciation functions, are the ones prescribed in Section 4.1.

Microeconomic actions (controls) As before, the microeconomic agents in the economy are assumed to control their consumption level c_t and their investment i_t.

Microeconomics dynamics The stock of consumer goods is driven by the following:

$$\dot{a}_t = r_t a_t - c_t - p_t i_t + F(k_t, p_t). \tag{4.11}$$

Notice that (4.11) depends explicitly on the interest rate of the economy, r_t. The stock of capital varies according to

$$\dot{k}_t = g(k_t, p_t) + i_t. \tag{4.12}$$

We notice that, introducing a central bank in the economy does not influence directly the dynamics of the capital accumulation, for the interest rate is not included in (4.12). Meanwhile, because it affects decisions of consumption and investment, the impact of r_t on the capital accumulation is encoded in the equilibrium condition of the model.

Macroeconomics actions (controls) The central bank intervenes in the economy by setting its interest rate, r_t.

Microeconomic preferences In this setting, we assume the instantaneous utility function of the microeconomic agents to depend also on the interest rate r_t. That is, $u = u(c_t, i_t, a_t, k_t, p_t, r_t)$. It amounts to

$$V(a, k, t) = \sup_{c_t, i_t} \int_t^{\infty} e^{-\alpha(s-t)} u(c_s, i_s, a_s, k_s, p_s, r_s) ds.$$

Equilibrium condition We assume the equilibrium condition to be given as in Section 4.1. In the context of this model, it is a market-clearing condition as well.

Optimal control problem The microeconomic agents' optimal control problem is described by:

$$V_t(a,k,t) - \alpha V(a,k,t) + H(a,k,p,r,V_a,V_k) = 0,$$

where the Hamiltonian H is given by

$$H(a,k,p,r,V_a,V_k) = \sup_{c,i}\left(V_a\dot{a} + V_k\dot{k} + u\right). \tag{4.13}$$

The optimal controls, given in feedback form, are determined by

$$\begin{cases} H_{q_a}(a_t,k_t,p_t,r_t,V_a,V_k) \;=\; r_t^* a_t - c_t^* - p_t i_t^* + F(k_t,p_t), \\ H_{q_k}(a_t,k_t,p_t,r_t,V_a,V_k) \;=\; g(k_t,p_t) + i_t^*, \end{cases} \tag{4.14}$$

where H is given as in (4.13).

We notice the microeconomic controls in (4.14) are distinct from those in (4.6). Moreover, a macroeconomic policy, when it exists, depends on the microeconomic optimal controls and conversely.

Transport of the agents' population As before, the dynamics of the microeconomic agents' and the optimal controls (4.14) yield a transport equation. Under the assumption of rationality, this equation governs the evolutions of the agents' distribution $\rho(a,k,t)$. We have:

$$\rho_t + ((r^* a_t - c^* - p_t i^* + F(k,p_t))\rho)_a + ((g(k,p_t) + i^*)\rho)_k = 0. \tag{4.15}$$

When coupled with the optimization problem of the agents, (4.15) leads to the following mean-field game system:

$$\begin{cases} V_t(a,k,t) \;-\; \alpha V(a,k,t) \;+\; H(a,k,p_t,r_t,V_a,V_k) = 0, \\ \rho_t + (H_{q_a}\rho)_a + (H_{q_k}\rho)_k = 0, \end{cases} \tag{4.16}$$

where H is given by (4.13).

Macroeconomics dynamics In the economy, the amount of consumer goods a_t that is neither consumed nor invested earns an interest rate r_t, set by the central bank. Therefore, the position of the central bank A_t is governed by the dynamics:

$$\dot{A}_t = -r_t \int a\rho(a,k,t)dadk. \tag{4.17}$$

Macroeconomic welfare The macroeconomic agent aims at controlling two quantities of the model. These are the price level of the economy, p_t, and the total wealth of the system, given by

$$W_t = \int a\rho(a,k,t) + p_t \int k\rho(a,k,t).$$

The central bank maximizes the welfare function of the economy, given by $U = U(A_t, W_t, p_t)$. The optimization problem faced by the central bank is formulated in terms of the intertemporal counterpart of U

$$\sup_{r_t} \int_t^\infty e^{-\beta(s-t)} U(A_s, W_s, p_s)ds. \tag{4.18}$$

When it exists, the optimal control r_t^* will be referred to as the *macroeconomic policy*.

The function U can be taken to be general enough to accommodate various formulations. For example, a central bank may be committed to keep the price level $p_t \equiv \psi$, or in a certain neighborhood of ψ, where $\psi > 0$ is a given constant, as in a typical inflation targeting regime. Alternatively, the goal of the central bank may be to sustain the economic growth.

Macroeconomic policy The macroeconomic policy is determined by the control problem (4.18), where the control r_t affects the macroeconomic state (A, ρ) through the dynamic comprising (4.17) and (4.16). Therefore, the optimization problem of the macroeconomic agent can be regarded as the control problem of an ordinary differential equation (4.17) and a system of partial differential equations (4.16).

4.3 Economic growth in the presence of trade imbalances

In this section, we propose a model of the economy in the presence of trade imbalances. In brief, it means that the market-clearing condition stated in (4.9) fails to hold. Trade imbalances can be introduced into the economy by various forces. Next, we distinguish between trade imbalances in the setting of a closed (single) economy and the case of international trade.

4.3.1 Single economy with a trade imbalance and regulated prices

Trade imbalance in the setting of a single economy can have multiple causes. For example, the price level, p_t, may be subjected to distortions - as in the case of a central planner - or follow a dynamic prescribed a priori. In what follows, we consider the setup of the economy described in Section 4.2, except for the equilibrium condition, which is here modified to take into account that the market fails to clear.

Unlike in the previous formulations, here, we do not assume the price level, p_t, to be determined in equilibrium. Instead, we suppose it is regulated by a central planner, or any other entity external to the economy. Because the price level is not determined by (4.9), we say that a *trade imbalance* appears in the system.

Equilibrium conditions Trade imbalances quantify the difference between the total investment in the economy and the production of capital goods. Given an extrinsic price level p_t, this quantity, denoted E_t, will be given by

$$E_t = \int i d\rho(a,k,t) - \int \Xi(k,p) d\rho(a,k,t).$$

The case $E_t > 0$ represents an excess of demand for capital goods, whereas $E_t < 0$ stands for an excess in supply. Therefore, for $E_t \neq 0$, the market does not clear.

Microeconomic state variables In this setting, the stock of consumer goods and the capital accumulation are governed, as before, by (4.11) and (4.12), respectively.

Microeconomic preferences Agents in the economy may have preferences over the trade imbalance E_t. These are encoded by the instantaneous utility function $u = u(c_t, i_t, a_t, k_t, p_t, r_t, E_t)$. Hence, the intertemporal counterpart of u becomes

$$V(a, k, t) = \sup_{c_t, i_t} \int_t^\infty e^{-\alpha(s-t)} u(c_s, i_s, a_s, k_s, p_s, r_s, E_s) ds. \quad (4.19)$$

Optimal control problem The microeconomic agents in the model solve the optimal control problem described by (4.11), (4.12) and (4.19). When the value function V has enough regularity, it solves

$$V_t - \alpha V + H(a, k, p_t, r, E, V_a, V_k) = 0,$$

where the Hamiltonian H is given by

$$H(a, k, p, r, E, V_a, V_k) = \sup_{c,i} \left(V_a \dot{a} + V_k \dot{k} + u \right). \quad (4.20)$$

The optimal controls associated with these problems are given in feedback form by the system

$$\begin{cases} H_{q_a}(a_t, k_t, p_t, r_t, E_t, V_a, V_k) = r_t a_t - c_t^* - p_t i_t^* + F(k_t, p_t), \\ H_{q_k}(a_t, k_t, p_t, r_t, E_t, V_a, V_k) = g(k_t, p_t) + i_t^*. \end{cases}$$

Transport of the agents' density Under the assumption of rationality, the evolution of the microeconomic agents' density ρ is governed by the following transport equation:

$$\rho_t + \left(H_{q_a}(a_t, k_t, p_t, r_t, E_t, V_a, V_k)\rho\right)_a + \left(H_{q_k}(a_t, k_t, p_t, r_t, E_t, V_a, V_k)\rho\right)_k = 0.$$

The mean-field game system associated with the single economy in the presence of trade imbalances is

$$\begin{cases} V_t(a, k, t) - \alpha V(a, k, t) + H(a, k, p_t, r, E, V_a, V_k) = 0, \\ \rho_t + \left(H_{q_a}\rho\right)_a + \left(H_{q_k}\rho\right)_k = 0, \end{cases} \quad (4.21)$$

where H is given by (4.20).

Macroeconomic preferences The preferences of the central bank are also affected by the trade imbalance. The welfare function becomes $U = U(A_t, W_t, p_t, E_t)$, where W_t is the total wealth

$$W_t = \int a\rho(a,k,t) + p_t \int k\rho(a,k,t).$$

As before, the optimal control problem of the macroeconomic agent is formulated in terms of the intertemporal counterpart of U

$$\max_{r_t} \int_t^{\infty} e^{-\beta(s-t)} U(A_s, W_s, p_s, E_s) ds. \tag{4.22}$$

Macroeconomic policies The macroeconomic policy is determined by the control problem (4.22), where the control r_t affects the macroeconomic state (A, ρ) through the dynamic comprising (4.17) and (4.21). Therefore, the optimization problem of the macroeconomic agent can be regarded as the control problem of an ordinary differential equation (4.17) and a system of partial differential equations (4.21).

4.3.2 Two countries economy with a trade imbalance

In this section, trade imbalance is introduced in the economy through an export and import flow. There are two countries in the economy, namely country 1 and country 2. The economy of the country i, with $i = 1, 2$, is defined as in Section 4.3.1, with the additional restriction

$$E_t^1 + E_t^2 = 0, \tag{4.23}$$

for every $t \geq 0$.

In this setting, given the interest rates of the two countries, the economy is completely characterized by two mean-field game systems, which we describe next.

The price levels in both countries may be identical, in which case it is determined by (4.23). Alternatively, one of the prices may be regulated. In that case, the price levels in countries 1 and 2 may differ, i.e., $p_t^1 \neq p_t^2$. Also, the production functions F^1 and F^2 can be different, reflecting the existence of distinct technologies in those countries. The associated MFG system is given by

$$\begin{cases} V_t^i(a^i, k^i, t) - \alpha V^i(a^i, k^i, t) + H^i(a^i, k^i, p_t^i, E^i, V_a^i, V_k^i) = 0, \\ \rho_t^i + \left(H_{q_a}^i \rho^i\right)_a + \left(H_{q_k}^i \rho^i\right)_k = 0, \end{cases} \tag{4.24}$$

for $i = 1, 2$.

A game between two macroeconomic agents Each of the central banks controls an interest rate r_t^i and seeks to maximize

$$\sup_{r_t^i} \int_t^\infty e^{-\beta(s-t)} U(A_s^i, W_s^i, p_s^i, E_s^i) ds,$$

where the state of the system is described by the mean-field games together with their assets dynamics

$$\dot{A}_t^i = -r_t^i \int a^i \rho(a^i, k^i, t) dadk. \tag{4.25}$$

Therefore, this formulation of the growth model in the presence of trade imbalances can be regarded as a non-cooperative game for two macroeconomic agents. Their dynamics are given by (4.25), under the constraints (4.24).

We notice that the MFG systems for both countries are coupled through (4.23).

4.4 Price impact in regulated markets

As in Section 4.3, we assume the price level of the economy to be driven by an extrinsic dynamic, instead of the balance condition in (4.9). This rigidity in the market entails distortions in the system called trade imbalances and encoded in the quantity E_t.

In the setup of Section 4.3, trade imbalances were implicitly taken into account by the agents. In what follows, we assume these distortions to impact explicitly the dynamics of the consumer goods' stock (for instance, as import taxes).

Price impact and the consumer goods' dynamics We assume that the price level of the economy is regulated extrinsically, which gives rise to trade imbalance, E_t. However, in the present setting, this quantity is monetized at a rate $\lambda(E_t)$. Then, the dynamics governing the stock of consumer goods becomes:

$$\dot{a}_t = c_t - (p_t + \lambda(E_t))\, i + F(k_t, p_t).$$

The rate $\lambda(E_t)$ can be regarded as a *price impact*. We assume that $\lambda(0) \equiv 0$; furthermore, $\lambda(q) \leq 0$ when $q \geq 0$ and $\lambda(q) \geq 0$ when $q \leq 0$. These assumptions on λ reflect intuitive notions behind the so-called *tatonnement* process. We assume this function λ is an important part of the model. Here, we do not consider any macroeconomic agent, although λ can be chosen by the government by imposing import taxes, for instance.

Optimal control problem As before, we are interested in

$$V(a, k, t) = \sup_{c,i} \int_t^\infty e^{-\alpha(s-t)} u(c_s, i_s, a_s, k_s, p_s, E_s) ds.$$

Provided certain regularity conditions hold, V solves

$$V_t - \alpha V + H(a, k, p_t, E, V_a, V_k) = 0,$$

where

$$H(a, k, p, E, V_a, V_k) = \sup_{c,i} \left(V_a \dot{a} + V_k \dot{k} + u \right). \tag{4.26}$$

The optimal controls, given in feedback form, for both the macroeconomic and the microeconomic agents are jointly determined by

$$\begin{cases} H_{q_a} = -c_t^* - (p_t + \lambda(E_t)) i_t^* + F(k_t, p_t), \\ H_{q_k} = g(k_t, p_t) + i_t^*, \end{cases} \tag{4.27}$$

where $H = H(a_t, k_t, p_t, E_t, V_a, V_k)$ is given by (4.26).

Next, we discuss the transport of the microeconomic agents' density, under the assumption of rationality.

Transport of the microeconomic agents' population The evolution of the agents' density ρ is governed by a transport equation of the form:

$$\begin{aligned} \rho_t(a,k,t) + \left[(-c-(p_t+\lambda(E))i + F(k,p_t))\,\rho\right]_a \\ + \left[(g(k,p_t)+i)\,\rho\right]_k = 0. \end{aligned} \tag{4.28}$$

By assuming rationality of the agents, i.e., the vector fields in (4.28) are evaluated at the optimal controls determined by (4.27), we get the following MFG:

$$\begin{cases} V_t(a,k,t) \;-\; \alpha V(a,k,t) \;+\; H(a,k,p_t,E,V_a,V_k) = 0, \\ \rho_t + \left(H_{q_a}\rho\right)_a + \left(H_{q_k}\rho\right)_k = 0, \end{cases}$$

where H is as in (4.26).

Bibliographical notes For a MFG formulation of the price impact model in the context of the Merton problem, see [87].

5

Mathematical analysis I - deterministic models

In this chapter, we develop the mathematical tools for the analysis of the problems put forth in the previous chapters. For the illustration of the main techniques, we examine the model outlined in Section 4.1, and we consider the finite horizon case. We address several mathematical questions. Firstly, we revisit the Verification Theorem (see Section 2.1) and prove that it holds in the setting of the growth model with heterogeneous agents described in Section 4.1. Next, we study monotonicity and concavity properties of the value function. After that, we prove the Pontryagin maximum principle, and establish the optimality of certain trajectories that solve suitable ODEs. Subsequently, we show the uniqueness of optimal trajectories. Finally, the chapter concludes with a discussion of the N-agent approximation.

5.1 A Verification Theorem

Here, we focus on the optimal control part of the model presented in Section 4.1 and state and prove a Verification Theorem. Since we work in the finite horizon setting, we start by fixing a terminal instant $T > 0$, and adjust the utility functional (4.3) accordingly. We consider the dynamics described by (4.1)-(4.2) and

$$V(a,k,t) = \sup_{c_t,i_t} \int_t^T e^{-\alpha(s-t)} u(c_s,i_s,a_s,k_s,p_s)ds + e^{-\alpha(T-t)} V_T(a_T,k_T), \tag{4.3'}$$

where $V_T : \mathbb{R}^2 \to \mathbb{R}$ is the terminal cost, which is a given data of the model.

Next, we show that a solution $U \in \mathcal{C}^1(\mathbb{R}^2 \times [t_0,T])$ of

$$\begin{cases} U_t - \alpha U + H(a,k,p,U_a,U_k) = 0 & \text{in } \mathbb{R}^3 \times [t_0,T] \\ U(a,k,T) = V_T(a,k) & \text{in } \mathbb{R}^3, \end{cases} \tag{5.1}$$

with

$$H(a,k,p,U_a,U_k) = \sup_{c,i} \left(U_a \dot{a} + U_k \dot{k} + u \right), \tag{5.2}$$

is the value function of the aforementioned optimal control problem, provided the instantaneous utility is strictly concave.

Theorem 5.1.1 (Verification Theorem)**.** *Let $U \in \mathcal{C}^1(\mathbb{R}^2 \times [t_0,T])$ be a solution of* (5.1) *and assume that the instantaneous utility function u is strictly concave. Then, U is the value function associated with the optimal control problem* (4.1)-(4.2) *and* (4.3').

Proof. As before (c.f. Section 2.1), we have

$$e^{-\alpha(T-t)} U(a_T,k_T,T) - U(a_t,k_t,t) = \int_t^T e^{-\alpha(s-t)} \left[U_s - \alpha U + U_a \dot{a} + U_k \dot{k} \right] ds. \tag{5.3}$$

In addition, (5.2) yields

$$U_s - \alpha U + U_a \dot{a} + U_k \dot{k} \leq -u. \tag{5.4}$$

Inequality (5.4) combined with (5.3) yields

$$U(a_t,k_t,t) \geq \int_t^T e^{-\alpha(s-t)} u ds + e^{-\alpha(T-t)} U(a_T,k_T,T),$$

hence

$$U(a,k,t) \geq V(a,k,t).$$

Because u is strictly concave, there exists a unique pair (c^*, i^*) so that

$$c^*, i^* \in \operatorname{argmax}\left(U_a\dot{a} + U_k\dot{k} + u\right).$$

Then, by Theorem 2.1.1, we have

$$U(a,k,t) = V(a,k,t).$$

□

5.2 Monotonicity and concavity

Here, we discuss two important qualitative properties of the value function, namely, its monotonicity and concavity. Since the value function represents the intertemporal utility of an agent, starting off with higher levels of utility at the initial instant, would allow her to sustain higher utility levels for future times as well. Indeed, an agent can trivially keep the previous investment and consumption levels and be at least not worse off. This explains the importance of the monotonicity.

Regarding concavity, it encodes the decreasing marginal increments of the utility. This is a cornerstone assumption in economic theory. Hence, this property for the intertemporal utility functional is critical in establishing the soundness of the model.

Throughout this section, we assume that p is fixed. We begin with a technical lemma:

Lemma 5.2.1. *Suppose the production function F is locally Lipschitz and non-decreasing in k, and the depreciation function g is locally Lipschitz in k. Fix initial states (a^1, k^1) and (a^2, k^2). Fix a control (c_t^*, i_t^*). Suppose $a^1 \leq a^2$ and $k^1 \leq k^2$. Let $(a^i(t), k^i(t))$ be a solution of*

$$\begin{cases} \dot{a}^i = -c_t^* - p_t i_t^* + F(k_t^i, p_t) \\ \dot{k}^i = g(k_t^i, p_t) + i_t^*, \end{cases}$$

with

$$\begin{cases} a^i_{t_0} &= a^i_0, \\ k^i_{t_0} &= k^i_0. \end{cases}$$

Then $a^1_t \leq a^2_t$ *and* $k^1_t \leq k^2_t$*, for all* $t \geq t_0$*.*

Proof. Because $k^1_0 \leq k^2_0$, the uniqueness of solutions of ordinary differential equations implies that $k^1_t \leq k^2_t$, for all $t \geq 0$. Then, by the monotonicity of F, we have $F(k^1_t, p_t) \leq F(k^2_t, p_t)$. Therefore, $\dot{a}^1_t \leq \dot{a}^2_t$ and so $a^1_t \leq a^2_t$, for all $t \geq t_0$. □

The first main result of this section is the monotonicity of the value function in the wealth variables (a, k).

Proposition 5.2.1. *Suppose that:*

i the instantaneous utility function u *is non-decreasing in the wealth variables* (a, k)*;*

ii the production function $F(k, p)$ *is non-decreasing in* k*;*

iii the depreciation function $g(k, p)$ *is locally Lipschitz in* k*.*

Then, the value function is non-decreasing in the (a, k) *variables.*

Proof. To simplify the proof, we suppose the existence of maximizers for (4.3'). This assumption is not essential, since the argument can be modified recurring to ϵ-optimal trajectories.

Fix initial states (a^1, k^1) and (a^2, k^2) such that $a^2 \geq a^1$ and $k^2 \geq k^1$. Let the optimal consumption and investment level for the first state be (c^*_t, i^*_t). Then, we have

$$V(a_1, k_1, t) = \int_t^T e^{-\alpha(s-t)} u(c^*_s, i^*_s, a^*_s, k^*_s) ds.$$

If we choose for the second state the same investment and consumption levels, this choice is clearly suboptimal. Hence, we have

$$V(a_2, k_2, t) \geq \int_t^T e^{-\alpha(s-t)} u(c^*_s, i^*_s, a_s, k_s) ds,$$

where (a_s, k_s) is driven by the controls (c_t^*, i_t^*), with initial state (a_2, k_2). According to Lemma 5.2.1, $a_t^* \leq a_t$ and $k_t^* \leq k_t$, for all $t \geq t_0$. Hence,

$$\begin{aligned} V(a_2, k_2, t) &\geq \int_t^T e^{-\alpha(s-t)} u(c_s^*, i_s^*, a_s, k_s) ds \\ &\geq \int_t^T e^{-\alpha(s-t)} u(c_s^*, i_s^*, a_s^*, k_s^*) ds \\ &= V(a_1, k_1, t). \end{aligned}$$

This proves the monotonicity of the value function. □

Next, we show that the value function is concave.

Proposition 5.2.2. *Suppose that:*

i u is concave in the microeconomic variables (c, i, a, k);

ii u is non-decreasing in the wealth variables (a, k);

iii the production function F is non-decreasing and concave in the capital variable k;

iv the depreciation function g is concave in the capital variable k.

Then $V(a, k, t)$ is concave in (a, k).

Proof. Consider two initial states (a^1, k^1) and (a^2, k^2). Suppose that the optimal consumption and investment levels for these two states are, respectively, (c_t^1, i_t^1) and (c_t^2, i_t^2). Consider the state $(a_t^\lambda, k_t^\lambda)$ given by

$$\begin{aligned} a_t^\lambda &= (1-\lambda)a_t^1 + \lambda a_t^2, \\ k_t^\lambda &= (1-\lambda)k_t^1 + \lambda k_t^2, \end{aligned}$$

with $\lambda \in (0, 1)$. Next, we show that

$$V(a_t^\lambda, k_t^\lambda, t) \geq (1-\lambda)V(a_t^1, k_t^1, t) + \lambda V(a_t^2, k_t^2, t).$$

Define

$$c_t^\lambda = (1-\lambda)c_t^1 + \lambda c_t^2,$$
$$i_t^\lambda = (1-\lambda)i_t^1 + \lambda i_t^2.$$

This strategy yields a sub-optimal trajectory for an agent with initial conditions $((1-\lambda)a^1 + \lambda a^2, (1-\lambda)k^1 + \lambda k^2)$. Such trajectory, $(\bar{a}_t, \bar{k}_t)$ is obtained by solving

$$\dot{\bar{a}}_t = -c_t^\lambda - p_t i_t^\lambda + F(\bar{k}_t, p_t),$$
$$\dot{\bar{k}}_t = i_t^\lambda + g(\bar{k}_t, p_t).$$

We claim that

$$\begin{cases} \bar{a}_t & \geq a_t^\lambda, \\ \bar{k}_t & \geq k_t^\lambda. \end{cases} \tag{5.5}$$

Using the concavity of the depreciation function g, we get:

$$\dot{\bar{k}}_t - g(\bar{k}_t, p_t) \geq \dot{k}_t^\lambda - g(k_t^\lambda, p_t).$$

Hence $\bar{k}_t \geq k_t^\lambda$ for all $t \geq 0$.

Next, we observe that, because of the concavity of the production function F, we have:

$$\begin{aligned} \dot{\bar{a}}_t - F(\bar{k}_t, p_t) &= -(1-\lambda)c_t^1 - \lambda c_t^2 - p_t((1-\lambda)i_t^1 + \lambda i_t^2) \\ &= (1-\lambda)(\dot{a}_t^1 - F(k_t^1, p_t)) + \lambda(\dot{a}_t^2 - F(k_t^2, p_t)) \\ &= \dot{a}_t^\lambda - (1-\lambda)F(k_t^1, p_t) - \lambda F(k_t^2, p_t) \\ &\geq \dot{a}_t^\lambda - F(k_t^\lambda, p_t) \geq \dot{a}_t^\lambda - F(\bar{k}_t, p_t). \end{aligned}$$

Then, $\bar{a}_t \geq a_t^\lambda$ and (5.5) is proven.

Furthermore, inequality (5.5) and matching initial conditions yield:

$$\begin{aligned}
V(a_t^\lambda, k_t^\lambda, t) &\geq \int_t^T e^{-\alpha(s-t)} u(c_s^\lambda, i_s^\lambda, \bar{a}_s, \bar{k}_s) ds \\
&\geq \int_t^T e^{-\alpha(s-t)} u(c_s^\lambda, i_s^\lambda, a_s^\lambda, k_s^\lambda) ds \\
&\geq (1-\lambda) \int_t^T e^{-\alpha(s-t)} u(c_s^1, i_s^1, a_s^1, k_s^1) ds \\
&+ \lambda \int_t^T e^{-\alpha(s-t)} u(c_s^2, i_s^2, a_s^2, k_s^2) ds \\
&= (1-\lambda) V(a_t^1, k_t^1, t) + \lambda V(a_t^2, k_t^2, t).
\end{aligned}$$

Hence, the proposition is established. □

5.3 Existence of optimal trajectories

In this section, we address the existence of the optimal trajectories for (4.3). Since the resources are limited in the economy, we suppose here that the consumption and investment levels of an individual agent are bounded by some natural constraints. That is,

$$|c| \leq \overline{C} \quad \text{and} \quad |i| \leq \overline{C}, \tag{5.6}$$

where c and i are, respectively, the consumption and investment levels and $\overline{C} > 0$ is some constant, chosen sufficiently large. As before, to make the presentation lighter, we work in the finite horizon setting.

Theorem 5.3.1. *Under the natural constraints* (5.6)*, and given a continuous price dynamics p_s, the problem*

$$\sup_{c_t, i_t} \int_t^T e^{-\alpha(s-t)} u(c_s, i_s, a_s, k_s, p_s) ds \tag{5.7}$$

admits an optimal trajectory.

Proof. We break the proof into two steps.
Step 1. We claim that the supremum in (5.7) is finite.

To prove this, we observe that under the natural constraints (5.6) all feasible trajectories, i.e., those satisfying (4.1), (4.2), are universally bounded. Namely, there exists a constant $\overline{M} > 0$ such that

$$|a_s|,\ |k_s| \leq \overline{M}. \tag{5.8}$$

Let $\bar{k}_s$ be the solution of the ODE

$$\begin{cases} \dot{\bar{k}}_s = \overline{C} + g(\bar{k}_s, p_s), \\ \bar{k}_0 = k_0. \end{cases}$$

Since $|i_s| \leq \overline{C}$, due to comparison principle for ODEs, we obtain that

$$k_s \leq \bar{k}_s \leq \|\bar{k}\|_\infty, \quad \text{for all} \quad s \in [t, T].$$

Similarly, if

$$\begin{cases} \dot{\underline{k}}_s = -\overline{C} + g(\underline{k}_s, p_s), \\ \underline{k}_0 = k_0, \end{cases}$$

we have

$$k_s \geq \underline{k}_s \geq -\|\underline{k}\|_\infty, \quad \text{for all} \quad s \in [t, T]. \tag{5.9}$$

This shows that the capital level has uniform bounds. Next, we address the consumer goods level. For that, let $\bar{a}_s$ be such that

$$\begin{cases} \dot{\bar{a}}_s = \overline{C} + p_s\overline{C} + F(\|\bar{k}\|_\infty, p_s), \\ \bar{a}_0 = a_0. \end{cases}$$

Because F is monotone increasing in the capital k and, moreover, $k_s \leq \bar{k}_s$, $|c_s| \leq \overline{C}$, $|i_s| \leq \overline{C}$, we obtain $\dot{a}_s \leq \dot{\bar{a}}_s$. Consequently,

$$a_s \leq \bar{a}_s \leq \|\bar{a}\|_\infty, \quad \text{for all} \quad s \in [t, T]. \tag{5.10}$$

Similarly, for $\underline{a}_s$ such that

$$\begin{cases} \dot{\underline{a}}_s = -\overline{C} - p_s\overline{C} + F(-\|\underline{k}\|_\infty, p_s), \\ \underline{a}_0 = a_0. \end{cases}$$

we have

$$a_s \geq \underline{a}_s \geq -\|\underline{a}\|_\infty, \quad \text{for all} \quad s \in [t,T]. \tag{5.11}$$

Combining (5.3), (5.9), (5.10) and (5.11), we get (5.8), where

$$\overline{M} = \max\{\|\bar{k}\|_\infty, \|\underline{k}\|_\infty, \|\bar{a}\|_\infty, \|\underline{a}\|_\infty\}.$$

Since u is continuous, we obtain

$$\overline{\Lambda} \doteq \sup\{u(c,i,a,k,p) \text{ s.t. } |c|,|i| \leq \overline{C}, |a|,|k| \leq \overline{M}, |p| \leq \|p\|_\infty\} < \infty.$$

Accordingly, for any control (c,i) such that (5.6) holds, we have:

$$\int_t^T e^{-\alpha(s-t)} u(c_s,i_s,a_s,k_s,p_s)ds \leq \overline{\Lambda} \int_t^T e^{-\alpha(s-t)} ds < \infty.$$

Therefore, the supremum in (5.7) is finite.

Step 2. An application of the direct method of calculus of variations.

Let (c^n,i^n,a^n,k^n) be a maximizing sequence for (5.7), i.e.,

$$\int_t^T e^{-\alpha(s-t)} u(c_s^n,i_s^n,a_s^n,k_s^n,p_s) \longrightarrow \sup_{c,i} \int_t^T e^{-\alpha(s-t)} u(c_s,i_s,a_s,k_s,p_s)$$

From (5.6), we have that

$$\|c^n\|_\infty, \|i^n\|_\infty \leq \overline{C}.$$

Consequently, we can extract a weak-* converging subsequence (which, by abuse of notation, we still denote by c^n, i^n). That is, there exist c, $i \in L^\infty([t,T])$ so that:

$$\int_t^T c_s^n \phi(s) + a_s^n \psi(s) ds \longrightarrow \int_t^T c_s \phi(s) + a_s \psi(s) ds, \tag{5.12}$$

for all ϕ, $\psi \in L^1([t,T])$.

Furthermore, we have that the sequences $\{a_s^n\}$, $\{k_s^n\}_n$ are uniformly bounded, due to (5.8), and equicontinuous, due to (5.6). Hence, by the Arzelà-Ascoli theorem, there is a further subsequence, which we again denote by a^n, k^n, and continuous functions a and k such that

$$a_s^n \to a_s \text{ and } k_s^n \to k_s, \quad \text{uniformly in} \quad [t,T]. \tag{5.13}$$

We have the identity

$$\begin{aligned} k_s^n &= \int_t^s i_\tau^n d\tau + \int_t^s g(k_\tau^n, p_\tau) d\tau \\ &= \int_t^T \chi_{[t,s]}(\tau) i_\tau^n d\tau + \int_t^s g(k_\tau^n, p_\tau) d\tau. \end{aligned}$$

Now, using the uniform convergence, in (5.13), and the weak-* convergence, in (5.12), and passing to the limit on both sides of the former equality, we get

$$\begin{aligned} k_s &= \int_t^T \chi_{[t,s]}(\tau) i_\tau d\tau + \int_t^s g(k_\tau, p_\tau) d\tau \\ &= \int_t^s i_\tau d\tau + \int_t^s g(k_\tau, p_\tau) d\tau. \end{aligned}$$

Therefore,

$$\dot{k}_s = i_s + g(k_s, p_s).$$

Similarly, we can prove that

$$\dot{a}_s = -c_s - p_s i_s + F(k_s, p_s).$$

Hence, (c_s, i_s, a_s, k_s) satisfy (4.1) and (4.2). This means that the quadruple (c_s, i_s, a_s, k_s) is a candidate for a maximizing trajectory. Note that the weak-* convergence and uniform convergence yield

$$\begin{cases} |c_s|,\ |i_s| & \leq \overline{C} \\ |a_s|,\ |k_s| & \leq \overline{M} \end{cases} \tag{5.14}$$

Using the concavity of u, we get that

$$\int_t^T e^{-\alpha(s-t)}u(c_s^n, i_s^n, a_s^n, k_s^n, p_s) \leq \int_t^T e^{-\alpha(s-t)}u(c_s, i_s, a_s, k_s, p_s) \tag{5.15}$$

$$+ \int_t^T e^{-\alpha(s-t)}D_c u(c_s, i_s, a_s, k_s, p_s)(c_s^n - c_s)ds$$

$$+ \int_t^T e^{-\alpha(s-t)}D_i u(c_s, i_s, a_s, k_s, p_s)(i_s^n - i_s)ds$$

$$+ \int_t^T e^{-\alpha(s-t)}D_a u(c_s, i_s, a_s, k_s, p_s)(a_s^n - a_s)ds$$

$$+ \int_t^T e^{-\alpha(s-t)}D_k u(c_s, i_s, a_s, k_s, p_s)(k_s^n - k_s)ds.$$

Since u is concave, it is locally Lipschitz. Hence, due to (5.14), we have that

$$|D_c u(c_s, i_s, a_s, k_s, p_s)| \leq \overline{L}$$
$$|D_i u(c_s, i_s, a_s, k_s, p_s)| \leq \overline{L}$$
$$|D_a u(c_s, i_s, a_s, k_s, p_s)| \leq \overline{L}$$
$$|D_k u(c_s, i_s, a_s, k_s, p_s)| \leq \overline{L},$$

for some constant $\overline{L} > 0$. This, combined with (5.15), (5.12) and (5.13), yields

$$\lim_{n\to\infty} \int_t^T e^{-\alpha(s-t)}u(c_s^n, i_s^n, a_s^n, k_s^n, p_s)ds$$
$$\leq \int_t^T e^{-\alpha(s-t)}u(c_s, i_s, a_s, k_s, p_s)ds.$$

Since the sequence (c^n, i^n, a^n, k^n) is maximizing, we conclude that the trajectory (c, i, a, k) is a maximizer. □

5.4 Optimal trajectories

As discussed previously, the Pontryagin maximum principle is a set of necessary conditions for optimality. However, under suitable concavity constraints, these necessary conditions are also sufficient for optimality.

Before proceeding, we present an auxiliary result.

Lemma 5.4.1. *Suppose the function x solves the backwards ordinary differential equation*

$$\begin{cases} -\dot{x}(s) = \alpha(s)x(s) + f(x(s), s), \\ x(T) = 0. \end{cases} \tag{5.16}$$

where $f \geq 0$. Then, we have that $x(s) \geq 0$ for $s \in [0, T]$.

Proof. To establish this lemma, we need to use the comparison principle for ODEs. Note that, because $f \geq 0$, the function $x \equiv 0$ is a subsolution for (5.16). Therefore, by the comparison principle, we have that $x(s) \geq 0$ for $s \in [0, T]$. □

Proposition 5.4.1. *Suppose that:*

i u is jointly concave in the (c, i, a, k) variables;

ii u is non-decreasing in the wealth variables (a, k);

iii the production function, F, is non-decreasing and concave in k, for p fixed;

iv the depreciation function, g, is concave in k.

Let (a_s^*, k_s^*) and (q_{as}^*, q_{ks}^*) be a solution of the Hamiltonian ODE

$$\begin{cases} \dot{a}_s = H_{q_a}(a_s, k_s, p_s, q_{as}, q_{ks}), \\ \dot{k}_s = H_{q_k}(a_s, k_s, p_s, q_{as}, q_{ks}), \\ -\frac{d}{ds}\left(e^{-\alpha(s-t)}q_{as}\right) = e^{-\alpha(s-t)}H_a(a_s, k_s, p_s, q_{as}, q_{ks}), \\ -\frac{d}{ds}\left(e^{-\alpha(s-t)}q_{ks}\right) = e^{-\alpha(s-t)}H_k(a_s, k_s, p_s, q_{as}, q_{ks}), \end{cases} \tag{5.17}$$

with initial-terminal conditions

$$\begin{cases} a_0^* = a_0, \\ k_0^* = k_0, \\ q_{aT}^* = 0, \\ q_{kT}^* = 0. \end{cases}$$

Then, $q_{as}^*,\ q_{ks}^* \geq 0$, *for* $0 \leq s \leq T$ *and* (a_s^*, k_s^*) *is an optimal trajectory corresponding to the optimal controls* (c_s^*, i_s^*), *that are jointly determined by*

$$\begin{cases} H_{q_a}(a_s, k_s, p_s, q_{as}, q_{ks}) \ = \ -\,c_s^* \ - \ p_s i_s^* \ + \ F(k_s, p_s), \\ H_{q_k}(a_s, k_s, p_s, q_{as}, q_{ks}) = \ g(k_s, p_s) \ + \ i_s^*. \end{cases} \tag{5.18}$$

Proof. The Hamiltonian H and the utility function u satisfy

$$\begin{aligned} u(c, i, a, k, p) \leq & H(a, k, p, q_a, q_k) \\ & - q_a(-c - pi + F(k, p)) - q_k(i + g(k, p)), \end{aligned}$$

where equality holds if and only if

$$\begin{cases} H_{q_a}(a, k, p, q_a, q_k) = -c - pi + F(k, p), \\ H_{q_k}(a, k, p, q_a, q_k) = i + g(k, p), \end{cases} \tag{5.19}$$

or, equivalently,

$$\begin{cases} D_c u(c, i, a, k, p) = q_a, \\ D_i u(c, i, a, k, p) = q_a p - q_k. \end{cases} \tag{5.20}$$

In addition, the derivatives of H and u satisfy the following relation:

$$\begin{cases} H_a(a, k, p, q_a, q_k) = D_a u(c, i, a, k, p), \\ H_k(a, k, p, q_a, q_k) = D_k u(c, i, a, k, p) + q_a D_k F(k, p) + q_k D_k g(k, p). \end{cases} \tag{5.21}$$

Let p_s be given, and (a_s, k_s) solve (5.17). In addition, let (c_s, i_s) be defined by (5.18). Since we assume that u is non-decreasing in c, we have that

$$q_{as}^* = D_c u(c_s^*, i_s^*, a_s^*, k_s^*, p_s) \geq 0.$$

Regarding the variable q^*_{ks}, because of (5.17) and (5.21), we have that

$$-\dot{q}_s = \left(D_k g(k^*_s, p_s) - \alpha\right) q_s + D_k u(c^*_s, i^*_s, a^*_s, k^*_s, p_s) + q^*_{as} D_k F(k^*_s, p_s).$$

Because $D_k u(c^*_s, i^*_s, a^*_s, k^*_s, p_s)$, $D_k F(k^*_s, p_s)$, and $q^*_{as} \geq 0$, Lemma 5.4.1 yields

$$q^*_{ks} \geq 0\ ,\ \text{ for } s \in [0, T].$$

Furthermore, the concavity of u yields

$$\begin{aligned}
&\int_t^\infty e^{-\alpha(s-t)} u(c_s, i_s, a_s, k_s, p_s) ds - \int_t^\infty e^{-\alpha(s-t)} u(c^*_s, i^*_s, a^*_s, k^*_s, p_s) ds \\
\leq &\int_t^\infty e^{-\alpha(s-t)} (D_c u(\ldots^*)(c_s - c^*_s) + D_i u(\ldots^*)(i_s - i^*_s) \\
&+ D_a u(\ldots^*)(a_s - a^*_s) + D_k u(\ldots^*)(k_s - k^*_s)) ds \\
= &\int_t^\infty e^{-\alpha(s-t)} (q^*_{as}(c_s - c^*_s) + (q^*_{as} p_s - q^*_{ks})(i_s - i^*_s) \\
&+ H_a(\ldots^*)(a_s - a^*_s) \\
&+ (H_k(\ldots^*) - q^*_{as} D_k F(k^*_s, p_s) - q^*_{ks} D_k g(k^*_s, p_s))(k_s - k^*_s)) ds \\
= &\int_t^\infty e^{-\alpha(s-t)} \Big(q^*_{as}(c_s + i_s p_s - c^*_s - i^*_s p_s - D_k F(k^*_s, p_s)(k_s - k^*_s)) \\
&- q^*_{ks}(i_s - i^*_s - D_k g(k^*_s, p_s)) \\
&+ H_a(\ldots^*)(a_s - a^*_s) + H_k(\ldots^*)(k_s - k^*_s) \Big) ds
\end{aligned}$$

$$
\begin{aligned}
&= \int_t^\infty - e^{-\alpha(s-t)} q^*_{as}(\dot a_s - \dot a^*_s) + \frac{d}{ds}\left(e^{-\alpha(s-t)} q^*_{as}\right)(a_s - a^*_s) \\
&\quad -e^{-\alpha(s-t)} q^*_{ks}(i_s - i^*_s) + \frac{d}{ds}\left(e^{-\alpha(s-t)} q^*_{ks}\right)(k_s - k^*_s) ds \\
&+ \int_t^\infty e^{-\alpha(s-t)} q^*_{as}(F(k_s, p_s) - F(k^*_s, p_s) - D_k F(k^*_s, p_s)(k_s - k^*_s)) \\
&\quad + e^{-\alpha(s-t)} q^*_{ks}(g(k_s, p_s) - g(k^*_s, p_s) - D_k g(k^*_s, p_s)(k_s - k^*_s)) ds \\
&\quad \le 0.
\end{aligned}
$$

The last inequality follows from a few elementary observations. Because of (5.17), and the matching initial and terminal conditions, the first integral vanishes. Moreover, the second integral is non-positive, due to the concavity of the production function F and the depreciation function g, combined with the inequalities $q^*_{as},\ q^*_{ks} \ge 0$. □

Remark 5.4.1. *The variables (q_a, q_k) are called adjoint variables.*

5.5 The Hamiltonian system and optimal trajectories

In the previous sections, we have shown that if the utility function u, the mixed production function F, and the depreciation function g are concave in the capital variable k, then the value function is monotone and concave. In addition, the solutions of the Pontryagin's maximum principle (or equivalently the Hamiltonian trajectories) are maximizers. Here, we show that the Hamiltonian system satisfies certain monotonicity conditions.

We consider the finite horizon problem without discount factor, to simplify the presentation.

In this case, the Hamiltonian system (5.17) takes the form

$$
\begin{cases}
\dot a_s = H_{q_a}(a_s, k_s, p_s, q_{as}, q_{ks}), \\
\dot k_s = H_{q_k}(a_s, k_s, p_s, q_{as}, q_{ks}), \\
\dot q_{as} = -H_a(a_s, k_s, p_s, q_{as}, q_{ks}), \\
\dot q_{ks} = -H_k(a_s, k_s, p_s, q_{as}, q_{ks}),
\end{cases}
$$

with initial conditions $a_0 = a(0)$, $k_0 = k(0)$ and terminal conditions $q_{aT} = q_a(T)$, $q_{kT} = q_k(T)$.

We introduce the operator

$$A \begin{bmatrix} q_a \\ q_k \\ a \\ k \end{bmatrix} = \begin{bmatrix} -\dot{a}_s + H_{q_a}(a_s, k_s, p_s, q_{as}, q_{ks}) \\ -\dot{k}_s + H_{q_k}(a_s, k_s, p_s, q_{as}, q_{ks}) \\ -\dot{q}_{as} - H_a(a_s, k_s, p_s, q_{as}, q_{ks}) \\ -\dot{q}_{ks} - H_k(a_s, k_s, p_s, q_{as}, q_{ks}) \end{bmatrix}. \tag{5.22}$$

We claim that this operator is monotone; let $\chi_i \doteq (q^i_{as}, q^i_{ks}, a^i_s, k^i_s)$, for $i = 1, 2$. An operator $A : H \to H$ is said to be monotone if:

$$\langle A\chi_1 - A\chi_2,\ \chi_1 - \chi_2 \rangle_H \geq 0,$$

for all trajectories $(q^1_{as}, q^1_{ks}, a^1_s, k^1_s)$ and $(q^2_{as}, q^2_{ks}, a^2_s, k^2_s)$ that satisfy the appropriate initial and terminal conditions and $q^1_{as}, q^1_{ks}, q^2_{as}, q^2_{ks} \geq 0$, where H is a Hilbert space and $\langle \cdot, \cdot \rangle_H$ is the inner product in H. For ease of notation, we drop the subscript H in what follows.

Proposition 5.5.1. *Suppose that*

i the utility function u is concave;

ii the production function F is concave and increasing in k;

iii the depreciation function g is concave in k.

Then, A given by (5.22) *is a monotone operator.*

Proof. Firstly, from Proposition 5.4.1 we have that $q^{1,2}_{as},\ q^{1,2}_{ks} \geq 0$.

Furthermore,

$$
\begin{aligned}
&\langle A\chi_1 - A\chi_2,\ \chi_1 - \chi_2\rangle \qquad (5.23)\\
&= \int_0^T (-\dot{a}_s^1 + \dot{a}_s^2)(q_{as}^1 - q_{as}^2) + (-\dot{k}_s^1 + \dot{k}_s^1)(q_{ks}^1 - q_{ks}^2)\\
&\quad + (-\dot{q}_{as}^1 + \dot{q}_{as}^2)(a_s^1 - a_s^2) + (-\dot{q}_{ks}^1 + \dot{q}_{ks}^2)(k_s^1 - k_s^2)ds\\
&\quad + \int_0^T \left(H_{q_a}(a_s^1, k_s^1, p_s, q_{as}^1, q_{ks}^2) - H_{q_a}(a_s^2, k_s^2, p_s, q_{as}^2, q_{ks}^2)\right)(q_{as}^1 - q_{as}^2)\\
+ &\quad \left(H_{q_k}(a_s^1, k_s^1, p_s, q_{as}^1, q_{ks}^2) - H_{q_k}(a_s^2, k_s^2, p_s, q_{as}^2, q_{ks}^2)\right)(q_{ks}^1 - q_{ks}^2)\\
+ &\quad \left(-D_a H(a_s^1, k_s^1, p_s, q_{as}^1, q_{ks}^2) + D_a H(a_s^2, k_s^2, p_s, q_{as}^2, q_{ks}^2)\right)(a_s^1 - a_s^2)\\
+ &\quad \left(-D_k H(a_s^1, k_s^1, p_s, q_{as}^1, q_{ks}^2) + D_k H(a_s^2, k_s^2, p_s, q_{as}^2, q_{ks}^2)\right)(k_s^1 - k_s^2)ds.
\end{aligned}
$$

For the first integral, note that

$$
\begin{aligned}
&\int_0^T (-\dot{a}_s^1 + \dot{a}_s^2)(q_{as}^1 - q_{as}^2) + (-\dot{k}_s^1 + \dot{k}_s^1)(q_{ks}^1 - q_{ks}^2)\\
&\quad + (-\dot{q}_{as}^1 + \dot{q}_{as}^2)(a_s^1 - a_s^2) + (-\dot{q}_{ks}^1 + \dot{q}_{ks}^2)(k_s^1 - k_s^2)ds\\
&\quad = \int_0^T \frac{d}{ds}\left((-a_s^1 + a_s^2)(q_{as}^1 - q_{as}^2) + (-k_s^1 + k_s^2)(q_{ks}^1 - q_{ks}^2)\right)\\
&\quad = 0,
\end{aligned}
$$

due to the initial and terminal conditions.

Next, we write the (a, k, q_a, q_k) variables in terms of the (c, i, a, k)

variables. For that, we use (5.19) and (5.21) in (5.23) and get that

$$
\begin{aligned}
&\langle A\chi_1 - A\chi_2,\ \chi_1 - \chi_2\rangle \\
&= \int_0^T \left(-c_s^1 - p_s i_s^1 + F(k_s^1, p_s) + c_s^2 + p_s i_s^2 - F(k_s^2, p_s)\right)(q_{as}^1 - q_{as}^2) \\
&\quad + \left(i_s^1 + g(k_s^1, p_s) - i_s^2 - g(k_s^2, p_s)\right)(q_{ks}^1 - q_{ks}^2) \\
&\quad + \left(-u_a(c_s^1, i_s^1, a_s^1, k_s^1, p_s) + u_a(c_s^2, i_s^2, a_s^2, k_s^2, p_s)\right)(a_s^1 - a_s^2) \\
&\quad + \left(-u_k(c_s^1, i_s^1, a_s^1, k_s^1, p_s) + u_k(c_s^2, i_s^2, a_s^2, k_s^2, p_s)\right)(k_s^1 - k_s^2) \\
&\quad - \left(q_{as}^1 F_k(k_s^1, p_s) + q_{ks}^1 g_k(k_s^1, p_s)\right)(k_s^1 - k_s^2) \\
&\qquad + \left(q_{as}^2 F_k(k_s^2, p_s) + q_{ks}^2 g_k(k_s^2, p_s)\right)(k_s^1 - k_s^2) ds.
\end{aligned}
$$

After rearranging the terms, and using the concavity of the functions F and g together with the positivity of $q_{as}^{1,2}$ and $q_{ks}^{1,2}$, we obtain the following inequality:

$$
\begin{aligned}
&\int_0^T -q_{as}^1 \left(F(k_s^2, p_s) - F(k_s^1, p_s) - D_k F(k_s^1, p_s)(k_s^2 - k_s^1)\right) \\
&\quad - q_{as}^2 \left(F(k_s^2, p_s) - F(k_s^2, p_s) - D_k F(k_s^2, p_s)(k_s^1 - k_s^2)\right) \\
&\quad - q_{ks}^1 \left(g(k_s^2, p_s) - g(k_s^1, p_s) - D_k g(k_s^1, p_s)(k_s^2 - k_s^1)\right) \\
&\quad - q_{ks}^2 \left(g(k_s^1, p_s) - g(k_s^2, p_s) - D_k g(k_s^2, p_s)(k_s^1 - k_s^2)\right) \\
&\quad - (q_{as}^1 - q_{as}^2)(c_s^1 - c_s^2) - (p_s q_{as}^1 - q_{ks}^1 - p_s q_{as}^2 + q_{ks}^2)(i_s^1 - i_s^2) \\
&\quad - \left(D_a u(c_s^1, i_s^1, a_s^1, k_s^1, p_s) - D_a u(c_s^2, i_s^2, a_s^2, k_s^2, p_s)\right)(a_s^1 - a_s^2) \\
&\quad - \left(D_k u(c_s^1, i_s^1, a_s^1, k_s^1, p_s) - D_k u(c_s^2, i_s^2, a_s^2, k_s^2, p_s)\right)(k_s^1 - k_s^2) ds \\
&\geq \int_0^T -(q_{as}^1 - q_{as}^2)(c_s^1 - c_s^2) - (p_s q_{as}^1 - q_{ks}^1 - p_s q_{as}^2 + q_{ks}^2)(i_s^1 - i_s^2) \\
&\quad - \left(D_a u(c_s^1, i_s^1, a_s^1, k_s^1, p_s) - D_a u(c_s^2, i_s^2, a_s^2, k_s^2, p_s)\right)(a_s^1 - a_s^2) \\
&\quad - \left(D_k u(c_s^1, i_s^1, a_s^1, k_s^1, p_s) - D_k u(c_s^2, i_s^2, a_s^2, k_s^2, p_s)\right)(k_s^1 - k_s^2) ds.
\end{aligned}
$$

By (5.20), the previous expression yields

$$\begin{aligned}\int_0^T &- \left(D_c u(c_s^1, i_s^1, a_s^1, k_s^1, p_s) - D_c u(c_s^2, i_s^2, a_s^2, k_s^2, p_s)\right)(c_s^1 - c_s^2) - \\ &- \left(D_i u(c_s^1, i_s^1, a_s^1, k_s^1, p_s) - D_i u(c_s^2, i_s^2, a_s^2, k_s^2, p_s)\right)(i_s^1 - i_s^2) \\ &- \left(D_a u(c_s^1, i_s^1, a_s^1, k_s^1, p_s) - D_a u(c_s^2, i_s^2, a_s^2, k_s^2, p_s)\right)(a_s^1 - a_s^2) \\ &- \left(D_k u(c_s^1, i_s^1, a_s^1, k_s^1, p_s) - D_k u(c_s^2, i_s^2, a_s^2, k_s^2, p_s)\right)(k_s^1 - k_s^2) ds \\ \geq 0,\end{aligned}$$

due to the concavity of u. Thus, we have

$$\langle A\chi_1 - A\chi_2,\ \chi_1 - \chi_2 \rangle \geq 0,$$

which finishes the proof. □

Remark 5.5.1. *Furthermore, let u be uniformly concave, i.e., the inequality*

$$\begin{aligned}&u(c^1, i^1, a^1, k^1, p) - u(c^2, i^2, a^2, k^2, p) \\ &- Du(c^2, i^2, a^2, k_2, p) \cdot (c^1 - c^2, i^1 - i^2, a^1 - a^2, k^1 - k^2) \\ &\leq -C \|(c^1 - c^2, i^1 - i^2, a_1 - a^2, k^1 - k^2)\|^2,\end{aligned}$$

holds for all $c^{1,2}, i^{1,2}, a^{1,2}, k^{1,2}$, where $C > 0$ is some constant. Then, we obtain a stronger inequality:

$$\begin{aligned}&\langle A\chi_1 - A\chi_2, \chi_1 - \chi_2 \rangle \\ &\quad \geq C \int_0^T (c_s^1 - c_s^2)^2 + (i_s^1 - i_s^2)^2 + (a_s^1 - a_s^2)^2 + (k_s^1 - k_s^2)^2 ds.\end{aligned} \tag{5.24}$$

If u is strictly concave, (5.24) *yields*

$$\langle A\chi_1 - A\chi_2, \chi_1 - \chi_2 \rangle > 0,$$

if $\chi_1 \neq \chi_2$.

As we show in the next corollary, this inequality allows us to prove the uniqueness of the optimal trajectories.

Corollary 5.5.1. *Suppose that*

i the utility function u is strictly concave;

ii the production function F is concave and increasing in k;

iii the depreciation function g is concave in k.

Then, there exists at most one optimal trajectory

$$\chi_s = (q_{as}, q_{ks}, a_s, k_s).$$

Proof. Suppose there are two optimal trajectories:

$$\chi_1 = (q^1_{as}, q^1_{ks}, a^1_s, k^1_s)$$

and

$$\chi_2 = (q^2_{as}, q^2_{ks}, a^2_s, k^2_s),$$

with the same initial-terminal conditions but $\chi_1 \neq \chi_2$. Then,

$$A\chi_1 = A\chi_2 = 0.$$

Hence,

$$\langle A\chi_1 - A\chi_2,\ \chi_1 - \chi_2 \rangle = 0,$$

which is a contradiction, so the optimal trajectory is unique. □

Theorem 5.5.1. *For arbitrary initial consumption and capital level, there exists a unique investment and consumption plan that maximizes the objective functional in* (4.3).

Consequently, from the classical optimal control theory, the value function is differentiable. Therefore, it solves the Hamilton-Jacobi equation in the classical sense.

Theorem 5.5.2. *The value function V defined by* (4.3) *is of class $\mathcal{C}^1$ and solves the Hamilton-Jacobi equation* (4.4) *in the classical sense.*

Proof. Since the optimal trajectories are unique for every initial consumption and capital level, we obtain that the value function is differentiable at all points. Hence, it satisfies the Hamilton-Jacobi equation in the classical sense rather than in the viscosity sense. Furthermore, since V is concave in (a, k) variables, it is continuously differentiable in (a, k) and the Hamilton-Jacobi equation yields that it is also continuously differentiable in the time variable t. □

5.6 N-agent approximation

Here, we examine the case of an economy with N-players. Then, the initial distribution of the agents has the form

$$\rho_0 = \frac{1}{N}\sum_{l=1}^{N}\delta_{(a^l,k^l)}, \tag{5.25}$$

where (a^l, k^l) are the initial consumer and capital goods levels for each of the agents, $l = 1, 2, \cdots, N$. According to our earlier analysis, if the price p_s is given, then under suitable conditions on the utility function u, production function F and depreciation function g, there exists a unique evolution ρ_s that satisfies the mean-field game equation (4.10).

If the initial distribution ρ_0 of is an average of N point masses, it is not clear whether or not ρ_s preserves this configuration for the future times, $s > t_0$. Heuristically, this means that the agents “do not split”. This is natural because an agent cannot use two different strategies at the same time.

Our aim, in this section, is to suggest a way to construct such a solution. Fix a price evolution p_s. For every agent $l \in \{1, 2, \cdots, N\}$, consider the Hamiltonian system

$$\begin{cases} \dot{a}^l_s = H_{q_a}(a^l_s, k^l_s, p_s, q^l_{as}, q^l_{ks}),\\ \dot{k}^l_s = H_{q_k}(a^l_s, k^l_s, p_s, q^l_{as}, q^l_{ks}),\\ -\frac{d}{ds}\left(e^{-\alpha(s-t)}q^l_{as}\right) = e^{-\alpha(s-t)}H_a(a^l_s, k^l_s, p_s, q^l_{as}, q^l_{ks}),\\ -\frac{d}{ds}\left(e^{-\alpha(s-t)}q^l_{ks}\right) = e^{-\alpha(s-t)}H_k(a^l_s, k^l_s, p_s, q^l_{as}, q^l_{ks}), \end{cases} \tag{5.26}$$

together with initial-terminal conditions

$$\begin{cases} a^l_t = a^l,\\ k^l_t = k^l,\\ q^l_{aT} = 0,\\ q^l_{aT} = 0. \end{cases}$$

As we have shown earlier, this system possesses a unique solution. Moreover, we have the following identities:

$$q^l_{as} = V_a(a^l_s, k^l_s, s), \quad q^l_{ks} = V_k(a^l_s, k^l_s, s).$$

Next, we construct a solution to (4.10).

Proposition 5.6.1. *Let* $(a^l, k^l) \in \mathbb{R}^2,\ l = 1, 2, \cdots, N,$ *be any initial consumption and capital goods levels and assume that* ρ_0 *is defined as in* (5.25)*. Furthermore, suppose that* (a_s^l, k_s^l) *is the unique solution of the system* (5.26)*. Define* ρ_s *as*

$$\rho_s = \frac{1}{N} \sum_{l=1}^{N} \delta_{(a_s^l, k_s^l)}.$$

Then, the couple (V, ρ) *solves* (4.10)*.*

Proof. The second equation in (4.10) is a transport equation with velocity field $(V_a(a, k, s), V_k(a, k, s))$. Since the transport equation is linear in the measure ρ, to prove that ρ_s, defined by (5.26), solves the transport equation, we just need to verify that ρ_s^l, given by

$$\rho_s^l = \delta_{(a_s^l, k_s^l)},$$

solves the transport equation for all $l \in \{1, 2, \cdots, N\}$. The trajectory (a_s^l, k_s^l) satisfies the ODE

$$\begin{cases} \dot{a}_s^l = H_{q_a}(a_s^l, k_s^l, p_s, V_a(a_s^l, k_s^l, s), V_a(a_s^l, k_s^l, s)) \\ \dot{k}_s^l = H_{q_k}(a_s^l, k_s^l, p_s, V_a(a_s^l, k_s^l, s), V_a(a_s^l, k_s^l, s)). \end{cases}$$

Hence, ρ_s^l satisfies the transport equation (see Section 2.2) and the proof is complete. □

All our previous analysis was based on the assumption that the price evolution p_s is given. To find the price p_s, we proceed as follows. We fix an arbitrary *smooth* evolution p_s of the price. Let the trajectories (a_s^l, k_s^l) be defined as in Proposition 5.6.1. Suppose $\tilde{p}_s$ is such that

$$\sum_{l=1}^{N} i_s^l = \sum_{l=1}^{N} \Xi(k_s^l, \tilde{p}_s), \tag{5.27}$$

where the control i_s^l is defined in the feedback form (4.6).

If $p_s = \tilde{p}_s$, then we have an existence result for the mean-field system (4.10), where we also prove the existence of the equilibrium price. Hence, the problem reduces to find a fixed point for the mapping

$$p_s \longrightarrow \tilde{p}_s.$$

Let us find a different form of the equation (5.27) to demonstrate the dependence of p_s of the left-hand side. We have

$$\sum_{l=1}^{N} i_s^l = \sum_{l=1}^{N} \Xi(k_s^l, \tilde{p}_s).$$

Note that

$$\begin{aligned} i_s^l &= \dot{k}_s^l - g(k_s^l, p_s) \\ &= H_{q_k}(a_s^l, k_s^l, V_a(a_s^l, k_s^l, s), V_k(a_s^l, k_s^l, s), p_s) - g(k_s^l, p_s). \end{aligned}$$

Hence,

$$\begin{aligned} &\sum_{l=1}^{N} \Big(H_{q_k}(a_s^l, k_s^l, V_a(a_s^l, k_s^l, s), V_k(a_s^l, k_s^l, s), p_s) \\ &- g(k_s^l, p_s) \Big) = \sum_{l=1}^{N} \Xi(k_s^l, \tilde{p}_s). \end{aligned}$$

Bibliographical notes For an account of the optimal control theory, we refer the reader to [126]. An introduction to the optimal control theory, targeting an undergraduate audience, can be found in the notes [57], by L. C. Evans.

The connection between deterministic optimal control and the theory of viscosity solutions of first-order Hamilton-Jacobi equations can be found in [17], [66] and [69].

Part II

Stochastic models

6

Second order MFG

6.1 Hamilton-Jacobi and Fokker-Plank equations

Before we proceed to the study of second order mean-field games, we put forward some preliminary material. In this section, we generalize several notions from the setting of deterministic optimal control to the stochastic setting. We start by addressing the Hamilton-Jacobi equation.

6.1.1 Hamilton-Jacobi equation

Consider a single agent whose state is determined by a point $x \in \mathbb{R}^d$. This agent can change its state by applying a control $v \in \mathbb{R}^d$. However, the agents are subject to independent random forces that are modeled by a white noise. To make matters precise, fix $T > 0$ and a filtered probability space, $\mathcal{P} = \Big(\Omega, \{\mathcal{F}_t\}_{0 \leq t \leq T}, P\Big)$, supporting a $d-$dimensional Brownian motion W_t. We also refer to $\mathcal{P}$ as a *stochastic basis*. Let $\sigma > 0$. In this simplified model, the trajectory of the agent is given by the stochastic differential equation (SDE)

$$\begin{cases} d\mathbf{x}_t = \mathbf{v}_t dt + \sigma dW_t, \\ \mathbf{x}_{t_0} = \mathbf{x}, \end{cases} \tag{6.1}$$

where $\mathbf{v}$ is a progressively measurable control with respect to the filtration $\mathcal{F}_t$, or simply $\mathcal{F}_t$-progressively measurable. That is, for each $0 \leq s \leq t$, the map $(s, \omega) \mapsto \mathbf{v}(s, \omega)$ is measurable with respect to $\mathcal{B}([t_0, t]) \times \mathcal{F}_t$.

Consider a Lagrangian $L : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$. By selecting the control v in a progressively measurable way, the agent seeks to maximize a functional cost given by

$$J(\mathbf{x}, \mathbf{v}; t) = \mathbb{E}^{\mathbf{x}} \left[\int_t^T L(\mathbf{x}_s, \mathbf{v}_s; m) ds \, + \, \Psi\left(\mathbf{x}_T\right) \right], \tag{6.2}$$

where m represents a quantity to be made precise later. In (6.2), $\mathbb{E}^x$ denotes the expectation operator, given that $\mathbf{x}_t = \mathbf{x}$. Furthermore, $\Psi : \mathbb{R}^d \to \mathbb{R}$ is the terminal cost of the system.

The Legendre transform of L is

$$H(x, p; m) = \sup_{v \in \mathbb{R}^d} \left(p \cdot v + L(x, v; m) \right). \tag{6.3}$$

We are interested in the value function of this problem, u, which is determined by

$$u(x, t) \, = \, \sup_{\mathbf{v}} J(\mathbf{x}, \mathbf{v}; t).$$

The function $u : \mathbb{R}^d \times [t_0, T] \to \mathbb{R}$ is called the value function associated with the (stochastic) optimal control problem (6.1)-(6.2). It is well known that if the value function u is regular enough (i.e., twice differentiable with respect to x and differentiable with respect to t), it is a solution of

$$u_t(x, t) \, + \, H(x, Du(x, t); m) \, + \, \frac{\operatorname{Tr} \sigma^T \sigma D^2 u(x, t)}{2} = 0, \tag{6.4}$$

equipped with the terminal condition

$$u(x, T) \, = \, \Psi(x). \tag{6.5}$$

In fact, the regularity requirement can be substantially relaxed: it is known that a viscosity solution of (6.4)-(6.5) is indeed the value function of the control problem described by (6.1)-(6.2). See [66].

In addition, the optimal control $\mathbf{v}^*$ is given in feedback form by

$$\mathbf{v}^* = H_p(x, Du(x,t); m).$$

The previous discussion can be made rigorous by means of a Verification Theorem. Before we proceed, we define the infinitesimal generator and state the Dynikin's formula.

Definition 6.1.1 (Infinitesimal generator)**.** *Let X_t be a Markov process, adapted to a fixed stochastic basis. The infinitesimal generator of the stochastic process X_t is the operator A, defined by*

$$Af(x_0,t_0) \doteq \lim_{t \downarrow t_0} \frac{\mathbb{E}^{x(t_0)=x_0}\left(f(X_t,t)\right) - f(x_0,t_0)}{t - t_0}. \tag{6.6}$$

The set of functions f, for which the limit in (6.6) *exists and is finite for all x, is called the domain of A and is denoted by $\mathcal{D}(A)$.*

Example 6.1.1 (Diffusion process)**.** *Consider the following SDE:*

$$d\mathbf{x}_s = h(\mathbf{x}_s, \mathbf{v}_s, s)ds + \sigma(\mathbf{x}_s, \mathbf{v}_s, s)dW_s, \tag{6.7}$$

where h and σ are given functions satisfying some growth and regularity conditions. Assume that $\mathbf{v}_s$ is a Markovian control. Then, (6.7) *is called a Markov diffusion. In this case, $A^{\mathbf{v}}$ is given by*

$$A^{\mathbf{v}}f(x,t) = \frac{\partial}{\partial t}f(x,t) + h \cdot f_x(x,t) + \frac{\operatorname{Tr}\sigma^T\sigma D^2 f(x,t)}{2}.$$

Before we state and prove a Verification Theorem, we present the Dynkin's formula.

Proposition 6.1.1 (Dynkin's formula)**.** *Let $\mathbf{x}_s$ be a Markov diffusion with infinitesimal generator A. Assume that $\mathbf{x}_{t_0} = \mathbf{x}$. If $f \in \mathcal{D}(A)$, then,*

$$\mathbb{E}^{(x,t_0)}\left(f(x_t,t)\right) - f(x,t_0) = \mathbb{E}^{(x,t_0)}\left(\int_{t_0}^{t} Af(x_s,s)ds\right),$$

for every $t \geq t_0$.

For a proof of Proposition 6.1.1, we refer the reader to [66]. Now, we proceed to the Verification Theorem.

Theorem 6.1.1 (Verification Theorem)**.** *Let w be a solution of* (6.4) *with the terminal condition* (6.5)*. Assume that w is differentiable with respect to the time variable and twice differentiable with respect to the space variable. Then, $w \geq u$. In addition, if the there exists $\mathbf{v}^*$ such that*

$$\mathbf{v}^* \in \operatorname{argmax}\left[A^{\mathbf{v}} w(\mathbf{x}_s^*, s) + L(\mathbf{x}_s^*, \mathbf{v}_s; m)\right],$$

we have $w = u$.

Proof. By applying the Dynkin's formula to w, we obtain

$$\mathbb{E}^{(x,t)}\left(w(x_T, T)\right) - w(x,t) = \mathbb{E}^{(x,t)}\left(\int_t^T A^v x(x_s, s) ds\right).$$

Therefore:

$$\begin{aligned} -w(x,t) &= \mathbb{E}^{(x,t)}\left(\int_t^T w_t + \mathbf{v}\cdot D_x w + \frac{\operatorname{Tr}\sigma^T \sigma D_x^2 w}{2} ds + u(x_T, T)\right) \\ &\leq \mathbb{E}^{(x,t)}\left(\int_t^T w_t + H + \frac{\operatorname{Tr}\sigma^T \sigma D_x^2 w}{2} - L ds + u(x_T, T)\right), \end{aligned} \tag{6.8}$$

where the inequality follows from the definition of H, in (6.3). Because w solves (6.4), (6.8) implies $w \geq u$.

Moreover, if there exists

$$\mathbf{v}^* \in \operatorname{argmax}\left[A^{\mathbf{v}} w(\mathbf{x}_s^*, s) + L(\mathbf{x}_s^*, \mathbf{v}_s; m)\right],$$

the inequality in (6.8) becomes an equality, that yields $w = u$. □

6.1.2 Fokker-Planck equation

In this section, we examine the Fokker-Planck equation. Consider a population of agents whose state is $x \in \mathbb{R}^d$. Assume further that the state of each agent in the population is governed by the stochastic differential equation in (6.1). Under the assumption of uncorrelated noise, the evolution of the population's density is determined by a

Fokker-Planck equation. To discuss the derivation of this equation, we depend once more on the notion of infinitesimal generator of a (Markov) process.

Let A be the generator of a Markov process $\mathbf{x}_t$. The formal adjoint of A, denoted by A^*, acts on functions in a suitable regularity class and is determined by the identity

$$\int_{\mathbb{R}^d} \phi(x) A f(x) dx = \int_{\mathbb{R}^d} f(x) A^* \phi(x) dx,$$

for every $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$.

Example 6.1.2 (Markov diffusions)**.** *The infinitesimal generator of a Markov diffusion is given in Example 6.1.1. It is:*

$$A^{\mathbf{v}}[f](x,t) = \frac{\partial}{\partial t} f(x,t) + h \cdot f_x(x,t) + \frac{\operatorname{Tr} \sigma^T \sigma D_x^2 f}{2}.$$

Therefore, A^* is given by

$$(A^{\mathbf{v}})^* [m] = -\frac{\partial}{\partial t} m - \operatorname{div}(hm) + \frac{\left(\left(\sigma^T \sigma\right)_{i,j} m \right)_{x_i x_j}}{2}.$$

A fundamental result (see, for example, [13]) states that the evolution of the population's density, given an initial configuration m_0, is described by the equation:

$$\begin{cases} A^*[m](x,t) = 0, \\ m(x,t_0) = m_0(x). \end{cases} \tag{6.9}$$

Example 6.1.2 builds upon (6.9) to yield the Fokker-Planck equation

$$m_t(x,t) + \operatorname{div}(hm(x,t)) = \frac{\left(\left(\sigma^T \sigma\right)_{i,j} m(x,t) \right)_{x_i x_j}}{2}. \tag{6.10}$$

6.2 Second-order mean-field games

Here, we combine several elements from Section 6.1 to derive a model second-order mean-field game system.

Consider a large population of agents whose state $\mathbf{x}_t \in \mathbb{R}^d$ is governed by (6.1). Assume further that each agent in this population faces the same optimization problem, given by (6.2). The Verification Theorem 6.1.1 shows that the solution of the Hamilton-Jacobi equation (6.4) is the value function. Moreover, the optimal control $\mathbf{v}^*$ is given in feedback form by

$$\mathbf{v}_t^* = H_p(\mathbf{x}, Du(\mathbf{x}, t); m).$$

On the other hand, the agents' population evolves according to (6.10). By setting $h \equiv \mathbf{v}$, we obtain

$$m_t(x,t) + \operatorname{div}(\mathbf{v}m(x,t)) = \frac{\left(\left(\sigma^T\sigma\right)_{i,j} m(x,t)\right)_{x_i x_j}}{2}.$$

Under the assumption of rationality of the agents, the population is driven by the optimal control $\mathbf{v}^*$; hence, it evolves according to

$$m_t(x,t) + \operatorname{div}(H_p(x, Du; m)m(x,t)) = \frac{\left(\left(\sigma^T\sigma\right)_{i,j} m(x,t)\right)_{x_i x_j}}{2}.$$

Therefore, the MFG system associated with (6.1)-(6.2) is:

$$\begin{cases} u_t + H(x, Du; m) + \frac{1}{2}\operatorname{Tr}\sigma^T\sigma D^2 u = 0, & (x,t) \in \mathbb{R}^d \times [t_0, T) \\ m_t + \operatorname{div}(H_p(x, Du; m)m) = \frac{\left(\left(\sigma^T\sigma\right)_{i,j} m\right)_{x_i x_j}}{2}, & (x,t) \in \mathbb{R}^d \times (t_0, T], \end{cases}$$

equipped with the initial-terminal conditions

$$\begin{cases} u(x,T) = u_T(x), \\ m(x,t_0) = m_0(x). \end{cases}$$

6.3 The non-linear adjoint method

Consider a (non-linear) differential operator $\mathcal{G} : Y \to Y$ and the corresponding homogeneous equation

$$\mathcal{G}[u](x) = 0. \tag{6.11}$$

It is possible to associate with (6.11) a linear equation that encodes important information about the solutions of (6.11). We start by considering then the linearized operator G of $\mathcal{G}$, determined by

$$\lim_{\|h\|_Y \to 0} \frac{\|\mathcal{G}(f+h) - \mathcal{G}(f) - G(f)h\|_Y}{\|h\|_Y} = 0,$$

and take its formal adjoint, in the L^2 sense:

$$\int \phi(x)\, G[v](x) dx = \int v(v)\, G^*[\phi](x).$$

The equation

$$G^*[m](x) = 0$$

is called the *adjoint* equation to (6.11), and m is called the adjoint variable. In what follows, we consider specific operators $\mathcal{G}$ and produce a few elementary results that illustrate the adjoint methods.

Set

$$\mathcal{G}[u](x,t) \doteq u_t + H(x, Du) + \Delta u,$$

where $H = H(x,p)$; i.e., $\mathcal{G}$ is the operator associated with the Hamilton-Jacobi equation (6.4). The adjoint equation is

$$m_t + \operatorname{div}(H_p(x, Du)m) = \Delta m. \tag{6.12}$$

Observe that the adjoint equation has maximum principle and preserves mass.

In the sequel, we equip (6.12) with appropriate initial conditions. This choice is arbitrary and motivated by the amount of information we can extract. We start by fixing an initial time $\tau \in [t_0, T)$ and a point x_0 in $\mathbb{R}^d$. Then, set

$$m(x,\tau) = \delta_{x_0}(x), \tag{6.13}$$

where $\delta_{x_0}(x)$ is the Dirac delta centered at x_0. The aforementioned choice leads to the next lemma.

Lemma 6.3.1 (Representation formula for u)**.** *Let u be a solution of* (6.4)-(6.5)*. Assume that m solves the adjoint equation* (6.12) *with* *initial condition* (6.13)*. Then*

$$\begin{aligned} u(x,t) &= \int_\tau^T \int_{\mathbb{R}^d} \left(H(x,Du) - H_p(x,Du)\cdot Du\right) m dx \\ &\quad + \int_{\mathbb{R}^d} u_T(x) m(x,T) dx. \end{aligned}$$

Proof. Multiply (6.4) by $-m$, (6.12) by u, sum them and integrate by parts to obtain

$$\frac{d}{dt}\int_{\mathbb{R}^d} umdx = \int_\tau^T \int_{\mathbb{R}^d} \left(H(x,Du) - H_p(x,Du)\cdot Du\right) m dx.$$

By integrating with respect to t, we get:

$$\begin{aligned} u(x_0,\tau) &= \int_\tau^T \int_{\mathbb{R}^d} \left(H(x,Du) - H_p(x,Du)\cdot Du\right) m dx \\ &\quad + \int_{\mathbb{R}^d} u_T(x) m(x,T) dx. \end{aligned}$$

Since x_0 and τ are chosen arbitrary, the former computation finishes the proof. □

Similar ideas yield a representation formula for the directional derivatives of the value function u.

Lemma 6.3.2 (Representation formula for u_ξ)**.** *Let u be a solution of* (6.4)-(6.5)*. Assume that m solves the adjoint equation* (6.12) *with initial condition* (6.13)*. Fix a direction ξ in $\mathbb{R}^d$. Then*

$$u_\xi(x,t) = \int_\tau^T \int_{\mathbb{R}^d} H_\xi(x,Du) dx dt + \int_{\mathbb{R}^d} (u_T)_\xi(x) m(x,T) dx$$

Proof. Differentiate (6.4) in the ξ direction and multiply it by m. Then, multiply (6.12) by u_ξ, sum them and integrate by parts to get:

$$\frac{d}{dt}\int_{\mathbb{R}^d} u_\xi(x,t) m(x,t) dx = \int_{\mathbb{R}^d} H_\xi(x,Du) m(x,t) dx.$$

Integrating the former equality with respect to t and noticing that x_0 and t are arbitrary, we obtain the result. □

Lemmas 6.3.1 and 6.3.2 give information about the values of u and u_ξ in a given point of the domain. These are determined by two quantities; the contribution of the energy H and an average of the terminal condition that depends, through $m(x,T)$, on the points of interest.

Next, we show that, under general assumptions on H, we can get uniform upper bounds for the solutions of (6.4).

Corollary 6.3.1 (Upper bounds for u)**.** *Let u be a solution of* (6.4)-(6.5). *Assume that m solves the adjoint equation* (6.12) *with initial condition* (6.13). *Assume further that there exists $C > 0$ so that*

$$H - H_p \cdot p \leq -C - CH. \tag{6.14}$$

Hence,

$$u(x,t) \leq \|u_T\|_{L^\infty(\mathbb{R}^d \times [t_0,T])},$$

for every $(x,t) \in \mathbb{R}^d \times [t_0,T]$.

Proof. It follows from Lemma 6.3.1, by using (6.14) and taking the supremum in the right-hand side of the representation formula. □

We conclude this section with a corollary about the Lipschitz regularity of the value function u. To simplify the presentation, we assume that H_x is uniformly bounded, i.e., there exists a constant $C > 0$ such that

$$|H_x(x,p)| \leq C.$$

Corollary 6.3.2 (Lipschitz regularity for u)**.** *Let u be a solution of* (6.4)-(6.5). *Assume that m solves the adjoint equation* (6.12) *with initial condition* (6.13). *Assume further that there exists $C > 0$ so that*

$$|H_x(x,p)| \leq C. \tag{6.15}$$

Hence,

$$Du \in L^\infty(\mathbb{R}^d \times [t_0,T]).$$

Proof. Lemma 6.3.2 yields:

$$u_\xi(x,t) = \int_\tau^T \int_{\mathbb{R}^d} H_x(x,Du)dxdt + \int_{\mathbb{R}^d} (u_T)_\xi(x)m(x,T)dx;$$

take absolute values on both sides of the previous equality and use (6.15) to conclude the proof. □

6.3.1 The adjoint structure of the MFG systems

A large collection of mean-field games enjoy an adjoint structure. That is, the Fokker-Planck equation is the adjoint of the linearized Hamilton-Jacobi equation. The system (6.2) is an important example where this holds. This fact has numerous consequences concerning the regularity of the solutions. However, the models in Chapter 4 and in the next chapter do not have this adjoint structure. In this particular case, however, by assuming a death rate for the population to be equal to the discount rate, the adjoint structure can be recovered.

The adjoint structure can also be regarded as a reflexion on the model of the rational expectations hypothesis as it means that the evolution of the distribution of the players is determined by the optimal feedback of the control problem each player faces.

Bibliographical notes Second-order mean-field games first appeared in the works of J.-M. Lasry and P.-L. Lions [119, 120]. Substantial material concerning these models can be found in the video-lectures by P.-L. Lions, [124, 125]. The notes by P. Cardaliaguet, [42] contain several results, explained in detail. For instance, the proof of existence of solutions based on Schauder's fixed point Theorem.

Smooth solutions for time-dependent, second-order MFG are studied in [45], for purely quadratic Hamiltonians. See also [46]. The quadratic case can be studied through a generalized Hopf-Cole transformation. However, general Hamiltonians require distinct techniques and are considered in [82, 81]. Systems with logarithmic nonlinearities are investigated in [78]. Additional results on regularity of solutions are reported in [79, 80]. The stationary case is first investigated in [118, 120], where weak solutions are discussed. Classical solutions for stationary second-order mean-field games are established in [77] and [85]. Recent developments are reported in [140]. Related problems are considered in [83] and [71]. The regularity theory for multi-population MFG is developed in [54].

For the adjoint method, we refer the reader to the original paper by L. C. Evans [59] as well as [151]. Further applications can be found in [37], [61], [38], [39], [36]. For applications of the adjoint methods to the MFG theory, see, for instance, [85], [77], [81] and [78].

7

Economic growth and MFG - stochastic setting

In the previous part of this book, we considered models where the states of the agents are deterministic. More realistic models of economic growth must include stochastic effects. These can be independent white noises, affecting each of the individual agents, or systemic effects. Here, we address the case of independent noise and revisit the models introduced in Chapter 4.

To make presentation simpler, we fix a filtered probability space $(\Omega, \{\mathcal{F}_s\}_{s\geq t_0}, P)$ throughout this chapter. We assume that all random quantities are adapted to $\{\mathcal{F}_s\}_{s\geq t_0}$. In addition, when a particular control process is said to be progressively measurable, we mean that it is progressively measurable with respect to $\{\mathcal{F}_s\}_{s\geq t_0}$.

7.1 A stochastic growth model with heterogeneous agents

We begin by revisiting the growth problem introduced in Section 4.1. We recall this model addresses the wealth and capital accumulation problems with a capital-dependent production. Here, the microeconomic states, macroeconomic variables, and the constitutive relations of the economy remain unchanged. The difference, in this section, is that each agent is affected by a white noise that is independent from the ones of the other agents.

Microeconomic agents At each instant t, a microeconomic agent is characterized by her level of consumer goods, a_t, and stock of capital, k_t.

Microeconomic actions (controls) We assume that each of the agents, at every instant t, controls her consumption and investment levels, c_t and i_t, respectively; these controls are progressively measurable processes.

Microeconomic dynamics In the present model, the evolution of the state variables of a typical agent is governed by two stochastic differential equations. The stock of consumer goods and the stock of capital are driven by

$$da_t = (-c_t - p_t i_t + F(k_t, p_t))dt + \sigma_a dB_t^a, \tag{7.1}$$

and

$$dk_t = (g(k_t, p_t) + i_t)\, dt + \sigma_k dB_t^k, \tag{7.2}$$

where B^a and B^k are adapted Brownian motions. We assume that distinct agents are affected by independent Brownian motions.

Macroeconomic variables The only macroeconomic variable of the model is the price level, p_t. This is a deterministic quantity in the model.

Constitutive relations The economy is provided with two constitutive relations, as discussed in Section 4.1. The first one is given by the production function, $F(k, p)$. The second one is determined by the depreciation function, $g = g(k, p)$.

Microeconomic preferences As before, agents in the economy face choices about their consumption and investment levels, and have preferences over their stocks of consumer goods and capital, and the price level of the economy. We assume that those preferences are represented by an instantaneous utility function $u = u(c_t, i_t, a_t, k_t, p_t)$. In this case, agents seek to maximize the expectation of the intertemporal utility functional

$$V(a,k,t) = \sup_{c_t, i_t} \mathbb{E}^{(a,k,t)} \int_t^\infty e^{-\alpha(s-t)} u(c_s, i_s, a_s, k_s, p_s) ds, \tag{7.3}$$

where $\mathbb{E}^{(a,k,t)}$ is the conditional expectation operator on $a_t = a,\ k_t = k$, the parameter $\alpha > 0$ is a discount rate, and we maximize with respect to all bounded progressively measurable controls (c_t, i_t).

Optimal control problem Each agent in the economy has an optimal control problem. The function $V(a,k,t)$, defined in (7.3), is called the *value function*.

If V has enough regularity, i.e., it is of class $\mathcal{C}^2$ in the space variable and of class $\mathcal{C}^1$ in the time variable, it solves the following Hamilton-Jacobi equation:

$$V_t - \alpha V \ + \ H(a,k,p,V_a,V_k) + \frac{\sigma_a^2}{2}\Delta_a V + \frac{\sigma_k^2}{2}\Delta_k V = 0, \tag{7.4}$$

where the Hamiltonian H is defined by:

$$H(a,k,p,V_a,V_k) = \sup_{c,i} \left((-c - pi + F)V_a + (g+i)V_k + u\right). \tag{7.5}$$

Moreover, the optimal controls c_t^* and i_t^* can be determined in feedback form $c_t^* = c^*(a_t, k_t, t)$, $i_t^* = i^*(a,k,t)$ from the equations

$$\left\{ \begin{array}{l} H_{q_a}(a,k,p_t,V_a(a,k,t),V_k(a,k,t)) \\ \qquad = -\, c^*(a,k,t) \, - \, p_t i^*(a,k,t) \, + \, F(k,p_t), \\ H_{q_k}(a,k,p_t,V_a(a,k,t),V_k(a,k,t)) = \, g(k,p_t) \, + \, i^*(a,k,t), \end{array} \right. \tag{7.6}$$

where $H = H(a,k,p,q_a,q_k)$ is given by (7.5).

Evolution of the agents' population As in the first-order case, we proceed by investigating the evolution of the agents' distribution in the space-state. In the stochastic setting, the agents' density is governed by a Fokker-Planck equation.

We assume that in (7.1)-(7.2) the controls are Markovian. This is the case for the optimal controls (c_t^*, i_t^*). In what follows, we present the infinitesimal generator associated with the stochastic process (7.1)-(7.2). The infinitesimal generator is a linear operator acting on regular enough functions. In some cases, it is important to emphasize the dependence of the infinitesimal generator A on the controls; when this is the case, we write A^{c_t,i_t}.

To avoid unnecessary complications regarding the domain of A, we define it on the space of smooth functions with compact support. Let $f \in \mathcal{C}_c^\infty$ and assume that c_t and i_t are Markovian controls, that is,

$$c_t \;=\; c(a_t, k_t),$$

and

$$i_t \;=\; i(a_t, k_t).$$

The infinitesimal generator A^{c_t,i_t} of the stochastic process (7.1)-(7.2) is:

$$\begin{aligned} A^{c,i} f(a,k) = (-c - pi + F)\,\frac{\partial f}{\partial a}(a,k) + (g+i)\,\frac{\partial f}{\partial k}(a,k) \\ + \frac{\sigma_a^2}{2}\Delta_a f(a,k) + \frac{\sigma_k^2}{2}\Delta_k f(a,k). \end{aligned}$$

The Fokker-Planck equation describing the evolution of the density ρ is given by

$$\rho_t(a,k,t) \;-\; \left(A^{c_t,i_t}\right)^* \rho(a,k,t) \;=\; 0,$$

where A^* is the formal adjoint of A in the L^2 sense; i.e.,

$$\int_{\mathbb{R}^2} \left(A^* \rho(a,k,t)\right) \phi(a,k,t) dadk \;=\; \int_{\mathbb{R}^2} \rho(a,k,t) \left(A\phi(a,k,t)\right) dadk,$$

for every $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^2)$. Therefore,

$$\rho_t + \left((-c - pi + F)\,\rho\right)_a + \left((g+i)\,\rho\right)_k = \frac{\sigma_a^2}{2}\Delta_a \rho + \frac{\sigma_k^2}{2}\Delta_k \rho. \tag{7.7}$$

Meanwhile, under the assumption of rationality, the agents act optimally, choosing the consumption and investment levels given by c_t^* and i_t^*, respectively. Therefore, (7.7) becomes

$$\rho_t + \left(H_{q_a}\rho\right)_a + \left(H_{q_k}\rho\right)_k = \frac{\sigma_a^2}{2}\Delta_a\rho + \frac{\sigma_k^2}{2}\Delta_k\rho. \tag{7.8}$$

Equilibrium conditions The model is equipped with an equilibrium condition that ensures a balance between aggregate investment and the production of capital goods. This condition couples the optimization problem, (7.4), with the transport of the agents' density, (7.8).

The equilibrium condition is:

$$\int i^*(a,k,t)d\rho(a,k,t) \; = \; \int \Xi(k,p)d\rho(a,k,t), \tag{7.9}$$

for every $t > 0$, where $i^*(a,k,t)$ is determined by (7.6).

As in the deterministic case, (7.9) determines the price level of the economy, p_t. Finally, we notice that (7.9) is also a market-clearing condition for the economy.

Observe here that (7.9) is consistent with the assumption that p_t is deterministic, for it does not depend on any random quantities.

A mean field game model The coupling of (7.4) and (7.8) leads to the system:

$$\begin{cases} V_t \; - \; \alpha V \; + \; H(a,k,p,V_a,V_k) + \frac{\sigma_a^2}{2}\Delta_a V + \frac{\sigma_k^2}{2}\Delta_k V = 0, \\ \rho_t + \left(H_{q_a}\rho\right)_a + \left(H_{q_k}\rho\right)_k = \frac{\sigma_a^2}{2}\Delta_a\rho + \frac{\sigma_k^2}{2}\Delta_k\rho, \end{cases}$$

together with the equilibrium condition (7.9), where $\rho(x,t_0) = \rho_0(x)$ is a given initial condition for the agents' distribution.

7.2 A stochastic growth model with a macroeconomic agent

Here, we introduce a macroeconomic agent into the previous setup, namely, a central bank. This agent controls the interest rate of the economy, the single macroeconomic variable of the model.

Microeconomic agents Microeconomic agents in the economy are characterized by their state variables: consumer goods a_t and capital k_t.

Macroeconomic agent There is one macroeconomic agent in the economy: the central bank. The state of the macroeconomic agent is determined by the amount A of its assets.

Macroeconomic variables As in the preceding model, the price level p_t is the single macroeconomic variable of the economy. Here, in addition to p_t, the economy has an interest rate, r_t. While the price level of the economy is determined by an equilibrium condition, the interest rate is set by the central bank.

Constitutive relations The economy has two constitutive relations, namely, the production and depreciation functions, F and g, respectively. These are prescribed as in Section 7.1.

Microeconomic actions (controls) Microeconomic agents in the economy control two quantities of the model, the consumption c_t and the investment level i_t.

Microeconomics dynamics In this model, the stock of consumer goods is governed by the stochastic differential equation:

$$da_t = (r_t a_t - c_t - p_t i_t + F(k_t, p_t))\, dt + \sigma_a dB_t^a,$$

where B_t^a is a scalar Brownian motion. The interest rate r_t impacts the accumulation of consumer goods.

The capital accumulation in the economy is governed by

$$dk_t = (g(k_t, p_t) + i_t)\, dt + \sigma_k dB_t^k,$$

where B_t^k is a one-dimensional Brownian motion, also adapted to a suitable stochastic basis.

As in Section 7.1, we assume independence of B_t^a and B_t^k among the population, i.e., there is no common noise in the model.

Macroeconomics dynamics The state of the central bank is determined by its total assets A_t and the distribution of agents $\rho(\cdot, t)$. The evolution of ρ is determined by (7.2). As suggested by the dynamics of consumer goods accumulation, the amount of consumer goods a_t that is not consumed neither turned into investment, earns the interest rate, r_t. Therefore, the assets of the central bank, A_t, evolve according to the following deterministic dynamics:

$$\dot{A}_t = -r_t \int a\rho(a,k,t)dadk. \tag{7.10}$$

Macroeconomics actions (controls) The central bank controls the interest rate of the economy, r_t.

Microeconomic preferences In the presence of a central bank, we suppose the instantaneous utility of the microeconomic agents to depend also on the interest rate r_t. I.e., $u = u(c_t, i_t, a_t, k_t, p_t, r_t)$. The intertemporal counterpart of u, denoted by V, is:

$$V(a,k,t) = \sup_{c,i} \int_t^\infty e^{-\alpha(s-t)} u(c_s, i_s, a_s, k_s, p_s, r_s)ds.$$

Equilibrium conditions We assume the equilibrium condition of the economy to be as in Section 7.1. Notice that it still represents a market-clearing condition.

Optimal control problem The optimal control problem faced by the microeconomic agents, in the present setting, is characterized by the Hamilton-Jacobi equation

$$V_t - \alpha V + H(a,k,p,r,V_a,V_k) + \frac{\sigma_a^2}{2}\Delta_a V + \frac{\sigma_k^2}{2}\Delta_k V = 0,$$

where the Hamiltonian H is:

$$H(a,k,p,r,V_a,V_k) = \sup_{c,i}\left((ra - c - pi + F)V_a + (g+i)V_k + u\right). \tag{7.11}$$

The optimal controls $c_t^* = c^*(a_t, k_t, t)$ and $i_t^* = i^*(a_t, k_t, t)$, associated with this problem, are given in feedback form by

$$\begin{cases} H_{q_a}(a, k_t, p_t, r_t, V_a(a,k,t), V_k(a,k,t) &= r_t a - c^* - p_t i^* + F(k, p_t), \\ H_{q_k}(a, k, p_t, r_t, V_a(a,k,t), V_k(a,k,t)) &= g(k, p_t) + i^*, \end{cases} \tag{7.12}$$

where H is defined as in (7.11).

Transport of the agents' population The state of the system evolves in this model according to two distinct equations. First, there is the Fokker-Planck equation, that describes the microeconomic agents' evolution. Further, the state of the central bank is characterized by (7.10).

The Fokker-Planck equation governing the evolution of the microeconomic agents' density is:

$$\rho_t + \left((r_t a - c^* - p i^* + F)\,\rho\right)_a + \left((g + i^*)\,\rho\right)_k = \frac{\sigma_a^2}{2}\Delta_a \rho + \frac{\sigma_k^2}{2}\Delta_k \rho. \tag{7.13}$$

As before, the assumption of rationality plays a major role here. It implies that the actual values of c_t, i_t and r_t are the optimal ones given in (7.12). Hence, (7.13) becomes

$$\rho_t + \left(H_{q_a}\rho\right)_a + \left(H_{q_k}\rho\right)_k = \frac{\sigma_a^2}{2}\Delta_a \rho + \frac{\sigma_k^2}{2}\Delta_k \rho.$$

A mean-field model of the economy Here, we formulate the growth model with a macroeconomic agent as a mean-field game system:

$$\begin{cases} V_t - \alpha V + H(a, k, p_t, r_t, V_a, V_k) + \frac{\sigma_a^2}{2}\Delta_a V + \frac{\sigma_k^2}{2}\Delta_k V = 0, \\ \rho_t + \left(H_{q_a}\rho\right)_a + \left(H_{q_k}\rho\right)_k = \frac{\sigma_a^2}{2}\Delta_a \rho + \frac{\sigma_k^2}{2}\Delta_k \rho, \end{cases}$$

where p_t is chosen so that (7.9) holds.

Macroeconomic welfare The macroeconomic agent has preferences over two quantities of the model. Typically, the central bank aims at controlling the price levels p_t while sustaining or increasing the wealth levels W_t in the economy. For example, the central bank may adopt an inflation target regime and operates through the interest rate to enforce the price to be within a particular range. Conversely, the priority of the macroeconomic agent may regard economic growth. The interest rate regulation seeks to ensure adequate levels of capital accumulation.

The objectives of the central bank are encoded in the welfare function of the economy, $U = U(A_t, W_t, p_t)$. As before, we formulate the optimization problem faced by the central bank in terms of the intertemporal counterpart of U:

$$\sup_r \int_t^\infty e^{-\beta(s-t)} U(A_s, W_s, p_s) ds. \tag{7.14}$$

Therefore, the central bank faces an infinite dimensional control problem, with payoff (7.14) and dynamics determined by (7.2), together with the equilibrium condition (7.9) and (7.10).

In the case an optimal control r_t^* exists, it is called a *macroeconomic policy*.

Bibliographical notes Economic models with second order effects were considered in [7] and [4]. The price formation models examined in [120, 30, 29, 35, 34, 25] fall outside the scope of this chapter but represent an important alternative approach. Stochastic MFG models also arise in power grid management, see for instance [15, 127].

8

Mathematical analysis II - stochastic models

In this chapter, we develop methods to study the stochastic models detailed in Chapter 7. We start with a Verification Theorem and proceed by investigating properties of concavity and monotonicity of solutions.

It is worth noticing that some of the material presented in Chapter 5 do not extend, at least in a straightforward manner, to the stochastic setting. This is the case of the N-agent approximation. Convergence of the N-agent approximation in the case of stationary second-order MFG, with ergodic costs, has been investigated in [118], [120] and [62]. See also [21].

For ease of presentation, this chapter focuses on the model of Section 7.1. In addition, we consider the finite horizon case, with a fixed terminal instant $T > 0$.

8.1 Verification Theorem

Here, we study the connection between solutions of

$$\begin{cases} W_t - \alpha W + H(a,k,p,W_a,W_k) + \frac{\sigma_a^2}{2}\Delta_a W + \frac{\sigma_k^2}{2}\Delta_k W = 0, \\ W(x,T) \; = \; u_T(x). \end{cases} \tag{8.1}$$

and the value function

$$V(a,k,t) = \sup_{c_t,i_t} \mathbb{E}^{(a,k,t)}\left[\int_t^T e^{-\alpha(s-t)}uds + e^{-\alpha(T-t)}u_T(x_T)\right]. \tag{8.2}$$

The Hamiltonian H in (8.1) is given by

$$H(a,k,p,W_a,W_k) = \sup_{c,i}\left(A^{(c,i)}W + u\right),$$

where $A^{(c,i)}$ is the infinitesimal generator of the Markov process (a_t,k_t).

Theorem 8.1.1 (Verification Theorem)**.** *Let W be a classical solution of* (8.1) *and assume that u is strictly concave. Then, W is the value function associated with the optimal control problem* (7.1)-(7.2) *and* (8.2)*.*

Proof. We start by verifying that, if W is a classical solution of (8.1), then

$$W(a,k,t) \geq V(a,k,t).$$

Assume that c and i are Markovian control processes. Using Dynkin's formula (Proposition 6.1.1), we obtain

$$\begin{aligned} &e^{-\alpha(T-t)}\mathbb{E}^{(a,k,t)}W(a_T,k_T,T) \; - \; W(a,k,t) \\ &\quad = \mathbb{E}^{(a,k,t)}\int_t^T e^{-\alpha(s-t)}\left[A^{c,i}W(a_s,k_s,s) - \alpha W(a_s,k_s,s)\right]ds, \end{aligned} \tag{8.3}$$

where A is the infinitesimal generator of the Markov process (a_t,k_t). Because W solves (8.1), we have that

$$A^{c,i}W(a_s,k_s,t) \; - \; \alpha W(a_s,k_s,s) \; + \; u \; \leq \; 0.$$

Therefore,

$$-W(a,k,t) \leq -\mathbb{E}^{(a,k,t)} \int_t^T e^{-\alpha(s-t)} u(c_s, i_s, a_s, k_s, p_s) ds \qquad (8.4)$$
$$- \, e^{-\alpha(T-t)} \mathbb{E}^{(a,k,t)} W(a_T, k_T, T).$$

To complete the proof, we notice that, because u is strictly concave, there exists a unique pair (c_t^*, i_t^*), such that

$$A^{c_t^*, i_t^*} W(a,k,t) + u(c_t^*, i_t^*, a_t, k_t, p_t) = 0. \qquad (8.5)$$

By using (8.5) in (8.3), one gets equality in (8.4). This establishes the Theorem. □

8.2 Monotonicity and convexity

Here, we investigate the monotonicity and concavity of the value function associated with the model considered in Section 7.1. Two reasons motivate our interest in those properties. First, they encode basic principles of the consumer's theory. Second, the instantaneous utility function u is assumed to be strictly concave and monotone. Therefore, it is important to understand whether or not its intertemporal counterpart inherits those properties. In the affirmative case, this ensures that preferences that can be represented by a suitable instantaneous utility function, can still be represented by an intertemporal functional.

We suppose throughout this section that the price, p_t, is given, deterministic and regular.

Monotonicity of the value function We begin with an auxiliary lemma that states that the dynamics (7.1)-(7.2) is order preserving, for each realization of the noise process.

Lemma 8.2.1. *Let $F(k,p)$ and $g(k,p)$ be continuous functions, locally Lipschitz in k, with F non-decreasing in the variable k. Fix two initial conditions, (a^1, k^1) and (a^2, k^2), with $a^1 \leq a^2$ and $k^1 \leq k^2$. Denote by (a^i, k^i) the solution of*

$$\begin{cases} da_t^i = (-c_t - p_t i_t + F(k_t^i, p_t))dt + \sigma_a dB_t^a, \\ dk_t^i = \left(g(k_t^i, p_t) + i_t\right) dt + \sigma_k dB_t^k, \end{cases}$$

equipped with initial conditions

$$\begin{cases} a^i_{t_0} &= a^i, \\ k^i_{t_0} &= k^i. \end{cases}$$

Then, $a^1_t \le a^2_t$ *and* $k^1_t \le k^2_t$*, for all* $t \ge t_0$*.*

Proof. We fix a realization of B^a_t and B^k_t. Thus, since g is locally Lipschitz in k,

$$dk^i_t = \big(g(k^i_t, p_t) + i_t\big)\, dt + \sigma_k dB^k_t$$

has a unique pathwise solution k^i with probability 1. The difference $k^1_t - k^2_t$ solves

$$d(k^1_t - k^2_t) = \big(g(k^1_t, p_t) - g(k^2_t, p_t)\big)\, dt.$$

Therefore, by an application of Gronwall's Lemma, we obtain $k^1_t \le k^2_t$ for all $t \ge t_0$, since $k^1 \le k^2$. Now, because F is locally Lipschitz in k, the SDE

$$da^i_t = (-c_t - p_t i_t + F(k^i_t, p_t))dt + \sigma_a dB^a_t$$

also has a unique pathwise solution a^i_t with probability 1. From the non-decreasing property of F in k, we infer that

$$F(k^1_t, p_t) \le F(k^2_t, p_t).$$

Consequently, for each path of B^a_t, we have

$$da^1_t \le da^2_t.$$

This concludes the proof. □

In the sequel, we present the proof of the monotonicity of the value function V.

Proposition 8.2.1 (Monotonicity of the value function)**.** *Let* $F(k,p)$ *and* $g(k,p)$ *be continuous functions, locally Lipschitz in* k*, with* $F(k,p)$ *non-decreasing in* k*. Assume that the instantaneous utility function* u *is non-decreasing in the wealth variables* a *and* k*. Then, the value function* V *is non-decreasing in* a *and* k*.*

Proof. We proceed by assuming the existence of maximizing trajectories, avoiding a more cumbersome argument, based on ϵ-optimality. As before, fix two initial conditions: (a^1, k^1) and (a^2, k^2), with $a^1 \leq a^2$ and $k^1 \leq k^2$. Denote by (c^j, i^j) the optimal control associated with the initial data (a^j, k^j). From the definition of the value function, it follows that

$$V(a^1, k^1, t) \;=\; \mathbb{E}^{(a^1,k^1,t)}\left[\int_t^T e^{-\alpha(s-t)} u(a_s^{1,1}, k_s^{1,1}, c_s^1, i_s^1) ds\right],$$

where $(a_s^{i,j}, k_s^{i,j})$ is the trajectory associated with the initial condition (a^i, k^i), driven by the control (c^j, i^j). Because the controls c^1 and i^1 are optimal for (a^1, k^1), they are suboptimal for (a^2, k^2). Therefore,

$$V(a^2, k^2, t) \;\geq\; \mathbb{E}^{(a^2,k^2,t)}\left[\int_t^T e^{-\alpha(s-t)} u(a_s^{2,1}, k_s^{2,1}, c_s^1, i_s^1) ds\right].$$

From Lemma 8.2.1, it follows that $a_s^{1,1} \leq a_s^{2,1}$ and $k_s^{1,1} \leq k_s^{2,1}$, for all $s \geq t_0$. This, combined with the fact that u is non-decreasing, yields:

$$\begin{aligned} V(a^1, k^1, t) &= \mathbb{E}^{(a^1,k^1,t)}\left[\int_t^T e^{-\alpha(s-t)} u(a_s^{1,1}, k_s^{1,1}, c_s^1, i_s^1) ds\right] \\ &\leq \mathbb{E}^{(a^2,k^2,t)}\left[\int_t^T e^{-\alpha(s-t)} u(a_s^{2,1}, k_s^{2,1}, c_s^1, i_s^1) ds\right] \leq V(a^2, k^2, t). \end{aligned}$$

□

The previous proposition ensures that the intertemporal utility function of the agents inherits the monotonicity of the instantaneous utility function. Next, we investigate the concavity of the value function.

Concavity of the value function The next proposition establishes the concavity of the value function, V.

Proposition 8.2.2 (Concavity of the value function)**.** *Let $F(k,p)$ and $g(k,p)$ be concave in k, and assume that $F(k,p)$ is a non-decreasing* *function, with respect to the variable k. Assume further that the instantaneous utility function u is non-decreasing in the wealth variables a and k and concave. Then, the value function V is concave in a and k.*

Proof. As before, we assume the existence of optimal trajectories and consider two distinct initial conditions: (a^1, k^1) and (a^2, k^2). For $\lambda \in (0,1)$, set

$$\begin{cases} a_s^\lambda \doteq (1-\lambda)a_s^1 + \lambda a_s^2, \\ k_s^\lambda \doteq (1-\lambda)k_s^1 + \lambda k_s^2. \end{cases}$$

Now, let (c_s^i, i_s^i) be the optimal controls associated with the initial conditions (a^i, k^i). Consider the convex combination

$$\begin{cases} c_s^\lambda \doteq (1-\lambda)c_s^1 + \lambda c_s^2, \\ i_s^\lambda \doteq (1-\lambda)i_s^1 + \lambda i_s^2. \end{cases}$$

Next, denote by $(\bar{a}_s, \bar{k}_s)$ the trajectory associated with $(c_s^\lambda, i_s^\lambda)$ and equipped with initial conditions $(a_s^\lambda, k_s^\lambda)$. It is clear that $(\bar{a}_s, \bar{k}_s)$ is suboptimal.

In addition, we have:

$$\begin{cases} d\bar{k}_t = \left(g(\bar{k}_t, p_t) + i^\lambda\right) dt + \sigma_k dB_t^k, \\ \bar{k}_0 = (1-\lambda)k^1 + \lambda k^2, \end{cases}$$

and, using the the concavity of g in k, we obtain

$$\begin{cases} dk_t^\lambda \leq \left(g(k_t^\lambda, p_t) + i^\lambda\right) dt + \sigma_k dB_t^k, \\ k_0^\lambda = (1-\lambda)k^1 + \lambda k^2. \end{cases}$$

Accordingly, by fixing a realization of the Brownian motion B_t^k, we have that $\bar{k}_t \geq k_t^\lambda$, for every $t \geq t_0$.

Moreover,

$$\begin{cases} d\bar{a}_t = \left(-c^\lambda - p_t i_t^\lambda + F(\bar{k}_t, p_t)\right) dt + \sigma_a dB_t^a, \\ \bar{a}_0 = (1-\lambda)a^1 + \lambda a^2. \end{cases}$$

Furthermore, using the concavity and the monotonicity of F, along with the fact that $\bar{k}_t \geq k_t^\lambda$, for every $t \geq t_0$, we conclude that

$$\begin{cases} da_t^\lambda \leq \left(-c^\lambda - p_t i_t^\lambda + F(\bar{k}_t, p_t)\right) dt + \sigma_a dB_t^a, \\ a_0^\lambda = (1-\lambda)a^1 + \lambda a^2. \end{cases}$$

Therefore, $\bar{a}_t \geq a_t^\lambda$, for every $t \geq t_0$. By combining this with the assumptions of monotonicity and convexity of u, we get

$$\begin{aligned} V(a^\lambda, k^\lambda, t) &\geq \mathbb{E}^{(\bar{a},\bar{k},t)} \left[\int_t^T e^{-\alpha(s-t)} u(c_s^\lambda, i_s^\lambda, \bar{a}_s, \bar{k}_s) ds \right] \\ &\geq \mathbb{E}^{(a^\lambda,k^\lambda,t)} \left[\int_t^T e^{-\alpha(s-t)} u(c_s^\lambda, i_s^\lambda, a_s^\lambda, k_s^\lambda) ds \right] \\ &\geq (1-\lambda)\mathbb{E}^{(a^1,k^1,t)} \left[\int_t^T e^{-\alpha(s-t)} u(c_s^1, i_s^1, a_s^1, k_s^1) ds \right] \\ &\quad + \lambda \mathbb{E}^{(a^2,k^2,t)} \left[\int_t^T e^{-\alpha(s-t)} u(c_s^2, i_s^2, a_s^2, k_s^2) ds \right] \\ &= (1-\lambda)V(a^1, k^1, t) + \lambda V(a^2, k^2, t). \end{aligned}$$

Hence, the proof is complete. □

Propositions 8.2.1 and 8.2.2 have an important implication regarding the intertemporal representation of the preference structure. These results ensure that the monotonicity and concavity of the instantaneous utility function are preserved under the intertemporal optimization problem of the agents.

8.3 A comparison result under regulated prices

We end this chapter by comparing the deterministic growth model, presented in Section 4.1, to its stochastic analog, introduced in Section 7.1. We assume that the price is regulated and given by the same deterministic price function p_t^*, both in the deterministic and the stochastic settings.

Denote by V^d the value function of the first-order MFG associated with the deterministic model. In the same fashion, let V^s denote the value function of the second-order MFG, associated with the stochastic model. We show that, for the given price level p_t^*, these functions are comparable, due to the concavity property of V^s.

Proposition 8.3.1 (Comparison)**.** *Let $F(k, p)$ and $g(k, p)$ be concave in k, and assume that $F(k, p)$ is a non-decreasing function, with respect to the variable k. Let the price level of the economy, p_t^*, be given. Assume further that the instantaneous utility function, u, is non-decreasing in the wealth variables a and k, and concave. Then,*

$$V^s \leq V^d.$$

Proof. We notice that V_d and V_s solve the following Hamilton-Jacobi equations:

$$V_t^d(a, k, t) - \alpha V^d(a, k, t) + H(a, k, p_t^*, V_a^d, V_k^d) = 0, \tag{8.6}$$

and

$$\begin{aligned} V_t^s(a, k, t) - \alpha V^s(a, k, t) + H(a, k, p_t^*, V_a^s, V_k^s) \\ = -\frac{\sigma_a^2}{2}\Delta_a V^s - \frac{\sigma_k^2}{2}\Delta_k V^s \\ \geq 0, \end{aligned} \tag{8.7}$$

where the inequality in (8.7) is implied by the concavity of V^s (c.f. Proposition 8.2.2). Notice that, because of (8.7), V^s is a subsolution of (8.6). A straightforward application of the comparison principle yields the desired result. □

When the price level of the economy is regulated, Proposition 8.3.1 states that, in the stochastic setting, an individual is at most as good as in the deterministic case. In other words, it shows that, for a given price level p_t^*, the welfare of an economy subject to noise is always below the welfare of a deterministic, more stable economic system.

Bibliographical notes The reader interested in the theory of second-order viscosity solutions is referred to [66] and [55]. Additional mathematical tools for economically motivated MFG are addressed in [4] and [7]. Mathematical methods for second-order MFG are discussed in detail in the surveys [42], [87], and [21].

9

Mean-field games with correlations

In the models presented in Chapter 7, we have assumed the independence of the Brownian motions affecting different agents. A further layer of complexity can be added, by allowing for correlated noise and systemic shocks. In the sequel, we examine a few preliminary notions of mean-field games in the presence of correlated noise. Then, we revisit the growth model from Section 4.1 taking into account the presence of systemic shocks.

For ease of presentation, we fix two distinct probability spaces throughout this chapter. Let $\tilde{\mathcal{P}} = (\tilde{\Omega}, \{\tilde{\mathcal{F}}_s\}_{s\geq t_0}, \tilde{P})$ be the stochastic basis associated with the driving noise. In addition, fix $\mathcal{P} = (\Omega, \{\mathcal{F}_s\}_{s\geq t_0}, P)$, the agents' space. Each $\omega \in \Omega$ represents an individual agent.

9.1 Mean-field games with correlated noise

9.1.1 General setup

As before, we consider a population of agents whose microscopic dynamics is governed by the stochastic differential equation:

$$\begin{cases} d\mathbf{x}_s = \mathbf{v}_s ds + \sqrt{2} dB_s, \\ \mathbf{x}(t) = \mathbf{x}, \end{cases} \tag{9.1}$$

where $\mathbf{v}_s$ is a $\tilde{\mathcal{F}}_s$-progressively measurable control and B_s is a d-dimensional Brownian motion, adapted to the filtration $\{\mathcal{F}_s\}_{s\geq t_0}$.

Fix $q > 1$ and let $\mathbf{X} \in L^q(\Omega, \mathbb{R}^d)$ be a random variable that assigns to each agent $\omega \in \Omega$ its position at time t, $\mathbf{X}_t(\omega)$. The probability distribution associated with the players' population is $\mathcal{L}(\mathbf{X})$, the law of $\mathbf{X}_s$. It is defined by

$$\mathbb{E}\phi(\mathbf{X}) = \int_{\mathbb{R}^d} \phi d\mathcal{L}(\mathbf{X}),$$

for any $\phi \in \mathcal{C}_c(\mathbb{R}^d)$.

Let $b : \mathbb{R}^d \times L^q(\Omega, \mathbb{R}^d) \times [t_0, T] \to \mathbb{R}^d$, be a Lipschitz vector field. We assume that b depends on the second coordinate only through the law of the random variable. That is, if $X, Y \in L^q(\Omega, \mathbb{R}^d)$ have the same law, we have $b(x, X, t) = b(x, Y, t)$. Next, let $\omega \in \Omega$ denote an arbitrary agent in the population. We suppose that at each instant $s > t_0$, the position of the agent ω in the state space is given by the random variable $\mathbf{X}_s(\omega)$, where $\mathbf{X}$ satisfies

$$\begin{cases} d\mathbf{X}_s(\omega) = b(\mathbf{X}_s(\omega), \mathbf{X}_s, s) ds + \sqrt{2} d\tilde{B}_s, \\ \mathbf{X}_{t_0} = X_0, \end{cases} \tag{9.2}$$

where $\tilde{B}_s$ is a d-dimensional Brownian motion adapted to $\{\tilde{\mathcal{F}}_s\}_{s\geq t_0}$, and $X_0 \in L^q(\Omega, \mathbb{R}^d)$ represents the initial states of the agents at time t_0. The random variable $\mathbf{X}$ contains all relevant information on the state of the agents.

Individuals in the population have preferences over their states, their controls, and the distributions of the remaining agents. As before, these are represented by a utility $u : \mathbb{R}^d \times \mathbb{R}^d \times L^q(\Omega, \mathbb{R}^d) \to \mathbb{R}$.

We assume $u(v, x, X)$ depends on the third coordinate only through the law of the random variable X. A typical agent in the population seeks to maximize

$$J_\alpha(\mathbf{v}, x, X; t) = \tilde{\mathbb{E}}^x \int_t^T e^{-\alpha(s-t)} u(\mathbf{v}_s, \mathbf{x}_s, \mathbf{X}_s) ds \qquad (9.3)$$
$$+ u_T(\mathbf{x}(T), \mathbf{X}(T)),$$

where u_T is a terminal cost, $\alpha > 0$ is a discount rate and $T > 0$ is a fixed terminal instant. The value function associated with the optimal control problem comprising of (9.1), (9.2) and (9.3) is

$$V(x, X, t) = \sup_{\mathbf{v}} J_\alpha(\mathbf{v}, x, X; t).$$

We aim at characterizing V as the solution of a partial differential equation. This is examined in the next section.

9.1.2 The master equation

We begin our discussion by introducing adequate notation for certain directional derivatives of functions of random variables. The notation $b \cdot D_X V(x, X, t)$ denotes the directional derivative of V in the direction b

$$b \cdot D_X V(x, X, t) = \frac{d}{d\epsilon} V(x, X + \delta b(x, X, t), t) \Big|_{\delta = 0}$$

Fix a standard unit vector e^i in $\mathbb{R}^d$. The directional first derivative operator in the direction e^i, denoted δ_i, is defined by

$$\delta_i V(x, X, t) \doteq \lim_{\epsilon \to 0} \frac{V(x, X + \epsilon e^i, t) - V(x, X, t)}{\epsilon}.$$

The directional second derivative operator in the direction e^i, δ_i^2, is given by

$$\delta_i^2 V(x, X, t) \doteq \frac{d^2}{d\epsilon^2} V(x, X + \epsilon e^i, t) \Big|_{\epsilon = 0}.$$

We consider the elliptic operator

$$\mathfrak{L} V(x, X, t) = \sum_{i=1}^d \delta_i^2 V(x, X, t) + 2 \sum_{i=1}^d \delta_i D_{x_i} V(x, X, t) \qquad (9.4)$$
$$+ \Delta_x V(x, X, t).$$

Next, we define the Hamiltonian $H : \mathbb{R}^d \times L^q(\Omega, \mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}$ by

$$H(x, X, p) = \sup_v \left(p \cdot v + u(v, x, X)\right).$$

Arguing as in Section 6.1, we see that, if b is such that (9.2) admits a unique solution X and V is a two-times Frechet differentiable function, solving the Hamilton-Jacobi equation

$$V_t - \alpha V + D_X V \cdot b + H(x, X, D_x V) + \mathfrak{L}V = 0, \tag{9.5}$$

where $\mathfrak{L}$ is the linear operator in (9.4), then V is the value function. Furthermore, the optimal vector field b^* is given by

$$b^*(x, \mathbf{X}, s) = D_p H(x, \mathbf{X}, D_x V(x, \mathbf{X}, s)). \tag{9.6}$$

Under the assumption of rationality, each agent in the population chooses the optimal vector field in (9.6). Therefore, by combining (9.5) with (9.6), we are lead to the mean-field equation in master form

$$\begin{aligned} &V_t(\mathbf{x}, \mathbf{X}, t) - \alpha V(\mathbf{x}, \mathbf{X}, t) \\ &\quad + D_X V(\mathbf{x}, \mathbf{X}, t) \cdot D_p H(\mathbf{X}(\omega), \mathbf{X}, D_x V(\mathbf{X}(\omega), \mathbf{X}, t)) \\ &\quad + H(\mathbf{X}(\omega), \mathbf{X}, D_x V(\mathbf{x}, \mathbf{X}, t)) + \mathfrak{L}V(\mathbf{x}, \mathbf{X}, t) \\ &= 0, \end{aligned} \tag{9.7}$$

together with the terminal condition

$$V(x, X, T) = u_T(x, X).$$

This is called the *master equation*. Notice that (9.7) encodes the information regarding the optimization problem faced by the agents and the evolution of the population's states.

9.2 A growth model with systemic shocks

Here, we put forward a growth model in the presence of systemic shocks.

Microeconomic agents A typical individual in the economy is characterized by her amounts a_t of consumer goods and k_t of capital goods.

At time $s \in [t_0, T]$, the position of the agent ω in the state space is given by the random variable $X_s(\omega) = (A_s(\omega), K_s(\omega))$.

Macroeconomic variables The only macroeconomic variable in the model is the (relative) price, p_t. This quantity is set in equilibrium. Here, p_t is no longer assumed to be deterministic, rather we suppose it is given in feedback form as a function of the state $X \in L^q(\Omega, \mathbb{R}^d)$ of the players, that is $p_t = p(X_t)$, for some function $p : L^q(\Omega, \mathbb{R}^d) \to \mathbb{R}$.

Constitutive relations There is a production function $F(k, p)$, that describes the technology of the economy. Moreover, a depreciation function $g(k, p)$ governs the depreciation of the capital stock, k_t.

Microeconomic actions (controls) In this model, the microeconomic agents control their levels of consumption and investment: c_t and i_t, respectively.

Microeconomic dynamics The state of the agents in the model is governed by two stochastic differential equations:

$$da_t = -c_t - p_t i_t + F(k_t, p_t) + \sigma^a dB_t^a,$$

and

$$dk_t = g(k_t, p_t) + i_t + \sigma^k dB_t^k,$$

where B_t^a and B_t^k are adapted Brownian motions.

In contrast with Chapter 7, here, we drop the assumption of independence of the Brownian motions impacting distinct agents. Therefore, the model is affected by systemic shocks, and the derivation of the associated MFG system, as in Chapter 7, is no longer valid.

We also prescribe the evolution of the random variable $\mathbf{X}_s$:

$$\begin{cases} d\mathbf{X}_s = b(a_s, k_s, \mathbf{X}_s, s)ds + \sqrt{2}d\tilde{B}_s, \\ \mathbf{X}_{t_0} = X, \end{cases}$$

where $\tilde{B}_s$ is an adapted Brownian motion and $b : \mathbb{R}^2 \times L^q(\Omega, \mathbb{R}^2) \times [t_0, T] \to \mathbb{R}^2$ is a Lipschitz vector field. As before, we denote by $\mathcal{L}(X)$ the law of the random variable X.

Microeconomic preferences Agents have preferences represented by an instantaneous utility function $u = u(c_t, i_t, a_t, k_t, X_t, p_t)$. The function u depends on the random variable X through its law. This dependence represents the preferences of the agents over the state of the population.

Its intertemporal counterpart is given by

$$V(a_t, k_t, X_t, t) = \sup_{c_t, i_t} \tilde{\mathbb{E}}^{(a,k,t)} \int_t^\infty e^{-\alpha(s-t)} u(c_s, i_s, a_s, k_s, X_s, p_s) ds.$$

Equilibrium condition The equilibrium condition of this model is

$$\int i d\mathcal{L}(\mathbf{X}_t)(a, k, t) = p_t \int \Xi d\mathcal{L}(\mathbf{X}_t)(a, k, t). \tag{9.8}$$

Master equation In contrast with Chapters 4 and 7, this model is characterized by the associated master equation. This is derived in the sequel.

First, notice that $V(a, k, X, t)$ solves

$$\begin{aligned} V_t(a, k, X, t) - \alpha V(a, k, X, t) &+ D_X V(a, k, X, t) \cdot b \\ &+ H(a, k, X, V_a, V_k, p(X, t)) \\ &+ \Delta_{(a,k)} V(a, k, X, t) \\ &+ \sum_{i=1}^{2} \delta_i^2 V(a, k, X, t) + 2\delta_1 D_a V(a, k, X, t) \\ &+ 2\delta_2 D_k V(a, k, X, t) = 0, \end{aligned} \tag{9.9}$$

where the Hamiltonian H is determined by:

$$H(a, k, X, V_a, V_k, p) = \max_{c_t, i_t} \left[A^{(c_t, i_t)} V + u \right],$$

where $A^{(c_t, i_t)}$ is the infinitesimal generator of the controlled Markov process (a_t, k_t), given the controls (c_t, i_t).

Because agents are rational, they act optimally, i.e., the vector field driving the evolution of the random variable $\mathbf{X}_s$ is

$$b(\mathbf{X}_s(\omega), \mathbf{X}_s, s) \;=\; D_p H(\mathbf{X}_s(\omega), \mathbf{X}_s, V_a, V_k, p(\mathbf{X}_s)).$$

Hence, (9.9) becomes

$$\begin{aligned} V_t - \alpha V + D_X V \cdot\, D_p H(\mathbf{X}_t(\omega), \mathbf{X}_t, V_a, V_k, p(\mathbf{X}_t)) \\ + H(a, k, \mathbf{X}_t, V_a, V_k, p(\mathbf{X}_t)) \\ = \Delta_{(a,k)} V + \sum_{i=1}^{2} \delta_i^2 V + 2\delta_1 D_a V + 2\delta_2 D_k V. \end{aligned} \tag{9.10}$$

Finally, the equilibrium condition (9.8) determines $p(X)$ implicitly as

$$p(\mathbf{X}_t) = \frac{\int \left(H_{q_k}(a, k, \mathbf{X}_t, V_a, V_k, p(\mathbf{X}_t)) - g(k, p(\mathbf{X}_t))\right) d\mathcal{L}(\mathbf{X}_t)(a, k, t)}{\int \Xi d\mathcal{L}(\mathbf{X}_t)(a, k, t)}, \tag{9.11}$$

where $(V_a, V_k) = (V_a(a, k, X, t), V_k(a, k, X, t))$.

Equations (9.10) and (9.11) contain all the relevant information on the model. First, it solves the optimization problem faced by the agents. Furthermore, it accounts for the evolution of this population, under the assumption of rationality, and takes into consideration the equilibrium condition (9.8).

Bibliographical notes For the master equation, we refer the reader to the video lectures by P.-L. Lions [124]. The paper by A. Bensoussan, J. Frehse and P. Yam [22] focuses on the master equation in the context of mean-field games and mean-field control. See also the monograph by the same authors [21]. A recent survey paper on the subject is [87]. Existence results can be found in [91, 68]. Recent developments regarding the existence of classical solutions for the master equation have been reported in [53]. In that paper, the authors consider the master equation without common noise.

For the probabilistic approach to mean-field game systems, we refer the reader to the work of R. Carmona and F. Delarue [49, 48] and R. Carmona, F. Delarue and D. Lacker [51] and R. Carmona and D. Lacker [52]. Finally, a recent approach to systemic risk is [143].

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